Divergence of Viscosity in Jammed Granular Materials: A Theoretical Approach

Koshiro Suzuki
Analysis Technology Development Center, Canon Inc.,
30-2 Shimomaruko 3-chome, Ohta-ku, Tokyo 146-8501, Japan

Hisao Hayakawa
Yukawa Institute for Theoretical Physics, Kyoto University,
Kitashirakawaoiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan

A theory for jammed granular materials is developed with the aid of a nonequilibrium steady-state distribution function. The approximate nonequilibrium steady-state distribution function is explicitly given in the weak dissipation regime by means of the relaxation time. The theory quantitatively agrees with the results of the molecular dynamics simulation on the critical behavior of the viscosity below the jamming point without introducing any fitting parameter.

PACS numbers: 05.20.Jj, 45.70.-n, 64.70.ps, 83.50.Ax

Introduction.– Description of granular rheology has been a long-term challenge for both science and technology. The problem extends to a vast range, from solid-like creep motion, gas-like, to liquid-like phenomena [1]. Similar to solid-liquid transitions, granular materials acquire rigidity when the density exceeds a critical value [2–5]. This phenomenon, referred to as the jamming transition, is universely observed in disordered materials such as colloidal suspensions [6], emulsions, and foams [7], as well as granular materials. The jamming transition and its relation to the glass transition have attracted much interest in the last two decades, and various aspects have been revealed [8–12]. In particular, characteristics in the vicinity of the jamming point, including the critical scaling behavior, have been extensively investigated by experiments and numerical simulations [2–4, 13–25]. It has been shown that the shear stress, the pressure, and the granular temperature can be expressed by scaling functions with exponents near $\varphi \sim \varphi_J$, where φ_J is the jamming transition density. The shear viscosity exhibits a form $\eta \sim (\varphi_J - \varphi)^{-\lambda}$ with $\lambda \approx 2$ for non-Browninan suspensions, foams, and emulsions [26–29], and a recent careful analysis demonstrated that λ is in the range between 1.67 and 2.55 [30]. It seems that the exponent λ for granular flows takes a larger value than that for suspensions [18, 19, 25, 31], although a value in the range mentioned above has been reported as well [17]. However, these studies are based on numerical simulations or phenomenologies without any foundation of a microscopic theory.

Even when we focus only on the flow properties below the jamming point $\varphi_{\rm J}$, which can be tracked back to Bagnold's work [32], we have not yet obtained a complete set in describing the rheological properties of dense granular flows. One of the remarkable achievements is the extension of the Boltzmann-Enskog kinetic theory to inelastic hard disks and spheres [33–38]. However, it has been recognized that the kinetic theory breaks down at densities with volume fraction $\varphi > \varphi_f = 0.49$ [39–42], since there exists correlated motions of grains. A modification of the kinetic theory has been proposed [43], but a microscopic theory is still absent.

Due to these situations, a microscopic liquid theory valid in the regime $\varphi_f < \varphi < \varphi_{\rm J}$ has been desired. One attempt to respond to this problem is the extension of the mode-coupling theory (MCT) [44] for dense granular liquids. MCT has been applied to granular systems driven by Gaussian noises [45–47]. It qualitatively predicts a liquid-glass transition, though the noise in granular systems is non-Gaussian in general [48–51]. MCT also has been applied to sheared dense granular systems [52–54]. There are three disadvantages of this approach: (i) the shift of φ is necessary to describe the divergence of η. (ii) Because of the shift of φ, the jamming transition is not correlated with the divergence of the first peak of the radial distribution function. (iii) It predicts a plateau in the density correlation function, which is not observed in experiments nor in simulations [55–57].

From these observations, it is crucial to obtain an explicit expression of the steady-state distribution function to construct a theory for dense granular liquids. For our purpose, we attempt to perform an expansion with respect to the dissipation to obtain an approximate explicit expression of the distribution function, valid in the weak dissipation regime for frictionless granular flows. Once the distribution is obtained explicitly, it is possible to calculate the steady-state average for arbitrary observables.

Microscopic starting equations.– We consider a three-dimensional system of N smooth granular particles of mass m and diameter d in a volume V subjected to stationary shearing characterized by the shear-rate tensor $\dot{\boldsymbol{\gamma}}$. We assume that each granular particle is a soft-sphere, and the contact force acts only on the normal direction. For a simple uniform shear with velocity along the x axis and its gradient along the y axis, the shear-rate tensor is $\dot{\gamma}_{\mu\nu} = \dot{\gamma}\delta_{\mu x}\delta_{\nu y}$ $(\mu, \nu = x, y, z)$ with a shear rate $\dot{\gamma}$. It is postulated that the applied shear induces a homogeneous streaming-velocity profile $\dot{\boldsymbol{\gamma}}\cdot\boldsymbol{r}$ at position $\boldsymbol{r}$, assuming that no heterogeneity such as shear banding exists.

Thus, the equation of motion is given by

$$\boldsymbol{p}_i = m\left(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{\gamma}}\cdot\boldsymbol{r}_i\right), \tag{1a}$$

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i^{(\mathrm{el})} + \boldsymbol{F}_i^{(\mathrm{vis})} - \dot{\boldsymbol{\gamma}}\cdot\boldsymbol{p}_i, \tag{1b}$$

where $\boldsymbol{F}_i^{(\mathrm{el})}$ and $\boldsymbol{F}_i^{(\mathrm{vis})}$ are, respectively, the elastic and the viscous contact forces acting on the grain i. Equation (1) is known as the Sllod equation, which is equivalent to Newton's equation of motion under a uniform shear [58].

The most essential feature of granular systems, in contrast to thermal systems, is that the steady state is determined by the balance between the viscous heating and the energy dissipation due to inelastic collisions. For sheared granular systems, this can be seen from the time derivative of the Hamiltonian, $\mathcal{H}(\boldsymbol{\Gamma}) = \sum_{i=1}^N \boldsymbol{p}_i^2/(2m) + \sum_{i,j}{}' u(r_{ij})$, where $u(r_{ij})$ is the inter-particle potential depending on $r_{ij} = |\boldsymbol{r}_i - \boldsymbol{r}_j|$, $\boldsymbol{\Gamma} = \{\boldsymbol{r}_i, \boldsymbol{p}_i\}_{i=1}^N$ is the phase-space coordinate, and $\sum_{i,j}{}'$ is the summation under the condition $i \neq j$. Then $\dot{\mathcal{H}}(\boldsymbol{\Gamma})$ satisfies

$$\dot{\mathcal{H}}(\boldsymbol{\Gamma}) = -\dot{\gamma} V \sigma_{xy}(\boldsymbol{\Gamma}) - 2\mathcal{R}(\boldsymbol{\Gamma}), \tag{2}$$

where

$$\sigma_{\mu\nu}(\boldsymbol{\Gamma}) = \frac{1}{V}\sum_{i=1}^N \left[\frac{p_{i,\mu}p_{i,\nu}}{m} + r_{i,\nu}\left(F_{i,\mu}^{(\mathrm{el})} + F_{i,\mu}^{(\mathrm{vis})}\right)\right] \tag{3}$$

is the stress tensor and

$$\mathcal{R}(\boldsymbol{\Gamma}) = -\frac{1}{2}\sum_{i=1}^N \dot{\boldsymbol{r}}_i \cdot \boldsymbol{F}_i^{(\mathrm{vis})} \tag{4}$$

corresponds to the Rayleigh's dissipation function [59, 60]. For granular systems with the interparticle dissipative force proportional to the relative velocity, it is impossible to reduce the dynamics as overdamped. For later analysis, we assume that the contact forces $\boldsymbol{F}_i^{(\mathrm{el})} = \sum_{j\neq i}\boldsymbol{F}_{ij}^{(\mathrm{el})}$ and $\boldsymbol{F}_i^{(\mathrm{vis})} = \sum_{j\neq i}\boldsymbol{F}_{ij}^{(\mathrm{vis})}$ are, respectively, given by $\boldsymbol{F}_{ij}^{(\mathrm{el})} = \kappa\,\Theta(d - r_{ij})(d - r_{ij})$ and $\boldsymbol{F}_{ij}^{(\mathrm{vis})} = -\zeta\,\Theta(d - r_{ij})(\boldsymbol{v}_{ij}\cdot\hat{\boldsymbol{r}}_{ij})\hat{\boldsymbol{r}}_{ij}$, where $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ otherwise, $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$, $\hat{\boldsymbol{r}}_{ij} = \boldsymbol{r}_{ij}/r_{ij}$, and $\boldsymbol{v}_{ij} = \boldsymbol{v}_i - \boldsymbol{v}_j$ with the velocity of i-th particle $\boldsymbol{v}_i$.

Steady-state distribution function.– To address the distribution function for the nonequilibrium steady state, we start from an equilibrium state at $t \to -\infty$ and evolve the system with shear and dissipation. Then, the system is expected to reach a steady state at $t = 0$. Although it is impossible to derive an exact solution of the Liouville equation, equivalent to Eq. (1), for the $6N$-dimensional distribution function, it is possible to obtain an approximate solution by perturbation, parallel to the method for the linearized Boltzmann equation [61]. In the perturbation for dense sheared granular systems, it is simple to obtain the leading-order eigenfrequency of the relaxation towards the steady state [60]. Hence, we attempt to speculate an *approximate* steady-state distribution, which we denote $\rho_{\mathrm{SS}}(\boldsymbol{\Gamma})$, by applying an approximation which explictly utilizes the relaxation time.

For this purpose, we start from a formal but *exact* expression for the distribution function [58],

$$\rho_{\mathrm{SS}}^{(\mathrm{ex})}(\boldsymbol{\Gamma}) = \exp\left[\int_{-\infty}^0 d\tau\,\Omega_{\mathrm{eq}}(\boldsymbol{\Gamma}(-\tau))\right]\rho_{\mathrm{eq}}(\boldsymbol{\Gamma}(-\infty)), \tag{5}$$

which is the steady-state solution of the Liouville equation. Here, $\Omega_{\mathrm{eq}}(\boldsymbol{\Gamma}) = \beta_{\mathrm{eq}}\dot{\mathcal{H}}(\boldsymbol{\Gamma}) - \Lambda(\boldsymbol{\Gamma}) = -\beta_{\mathrm{eq}}\left[\dot{\gamma}V\sigma_{xy}(\boldsymbol{\Gamma}) + 2\mathcal{R}(\boldsymbol{\Gamma})\right] - \Lambda(\boldsymbol{\Gamma})$ is the work function for $\rho_{\mathrm{eq}}(\boldsymbol{\Gamma}) = e^{-\beta_{\mathrm{eq}}\mathcal{H}(\boldsymbol{\Gamma})}/\int d\boldsymbol{\Gamma}\, e^{-\beta_{\mathrm{eq}}\mathcal{H}(\boldsymbol{\Gamma})}$ at temperature $T_{\mathrm{eq}} = \beta_{\mathrm{eq}}^{-1}$, where $\Lambda(\boldsymbol{\Gamma}) = (\partial/\partial\boldsymbol{\Gamma})\cdot\dot{\boldsymbol{\Gamma}}$ is the phase-space volume contraction. We approximate Eq. (5) by introducing the relaxation time τ_{rel} as

$$\exp\left[\int_{-\infty}^0 d\tau\,\Omega_{\mathrm{eq}}(\boldsymbol{\Gamma}(-\tau))\right] \approx e^{\tau_{\mathrm{rel}}\Omega_{\mathrm{SS}}(\boldsymbol{\Gamma})}, \tag{6}$$

which can be validated in the perturbation expansion of the Liouville equation around the canonical distribution [60]. In the perturbation, we non-dimensionalize all the quantities, where the units of mass, length, and time are chosen as m, d, and $\sqrt{m/\kappa}$, and introduce $\epsilon \equiv \zeta/\sqrt{\kappa m} \ll 1$ as a perturbation parameter, which is related to the restitution coefficient e as $\epsilon \approx \sqrt{2}(1-e)/\pi$ for $e \approx 1$. We attach a star $*$ to the non-dimensionalized quantites, e.g. $t^* = t\sqrt{\kappa/m}$. Furthermore, we perform a scaling which leaves the steady-state temperature T_{SS}, which is the ensemble average of $\sum_{i=1}^N \boldsymbol{p}_i^2/(3Nm)$ at the steady state in the dimensional unit, to be independent of ϵ. This indicates that the granular fluid keeps its motion in the limit $\epsilon \to 0$. From dimensional analysis, T_{SS} satisfies $T_{\mathrm{SS}} \sim m^3 d^2\dot{\gamma}^4/\zeta^2$, which leads to $T_{\mathrm{SS}}^* \sim \epsilon^{-2}\dot{\gamma}^{*4}$. Hence, $\dot{\gamma}^*$ should satisfy $\dot{\gamma}^* \sim \epsilon^{1/2}$. We introduce a scaled shear rate $\tilde{\dot{\gamma}}$ as $\dot{\gamma}^* = \epsilon^{1/2}\tilde{\dot{\gamma}}$, where $\tilde{\dot{\gamma}}$ is independent of ϵ. We attach a tilde to the scaled quantities. The relaxation time τ_{rel} is evaluated from the eigenfrequency of the perturbation expansion as

$$\tau_{\mathrm{rel}} = \left[\frac{2\sqrt{\pi}}{3}\epsilon\,\omega_E(T_{\mathrm{SS}})\right]^{-1} \tag{7}$$

in the hard-core limit [60], where $\omega_E(T) = 4\sqrt{\pi}\,n\sqrt{T/m}\,g_0(\varphi)d^2$ is the Enskog frequency of collisions and $g_0(\varphi)$ is the first-peak value of the radial distribution function. In Eq. (6), we have also introduced

$$\Omega_{\mathrm{SS}}(\boldsymbol{\Gamma}) = -\beta_{\mathrm{SS}}\left[\dot{\gamma}V\sigma_{xy}^{(\mathrm{el})}(\boldsymbol{\Gamma}) + 2\Delta\mathcal{R}_{\mathrm{SS}}^{(1)}(\boldsymbol{\Gamma})\right], \tag{8}$$

where $\sigma_{xy}^{(\mathrm{el})}$ and $\Delta R_{\mathrm{SS}}^{(1)}$ are respectively given by [60]

$$\sigma_{xy}^{(\mathrm{el})}(\boldsymbol{\Gamma}) = \frac{1}{V}\sum_{i=1}^N\left[\frac{p_{i,x}p_{i,y}}{m} + y_i F_{i,x}^{(\mathrm{el})}\right], \tag{9}$$

$$\Delta\mathcal{R}_{\mathrm{SS}}^{(1)}(\boldsymbol{\Gamma}) = \mathcal{R}^{(1)}(\boldsymbol{\Gamma}) + \frac{T_{\mathrm{SS}}}{2}\Lambda(\boldsymbol{\Gamma}), \tag{10}$$

$$\mathcal{R}^{(1)}(\boldsymbol{\Gamma}) = \frac{\zeta}{4}\sum_{i,j}{}'\left(\frac{\boldsymbol{p}_{ij}}{m}\cdot\hat{\boldsymbol{r}}_{ij}\right)^2\Theta(d - r_{ij}). \tag{11}$$

Here, we ignore the contribution from the viscous shear stress, which is a higher-order correction in the limit $\epsilon \to 0$. To summarize, we obtain

$$\rho_{\rm SS}(\mathbf{\Gamma}) = \frac{e^{-I_{\rm SS}(\mathbf{\Gamma})}}{\int d\mathbf{\Gamma} e^{-I_{\rm SS}(\mathbf{\Gamma})}}, \tag{12}$$

where $I_{\rm SS}(\mathbf{\Gamma}) = \beta_{\rm SS}^* \mathcal{H}^*(\mathbf{\Gamma}) - \tilde{\tau}_{\rm rel}\tilde{\Omega}_{\rm SS}(\mathbf{\Gamma})$ with $\tilde{\Omega}_{\rm SS}(\mathbf{\Gamma}) = -\beta_{\rm SS}^*\left[\dot{\tilde{\gamma}} V^* \tilde{\sigma}_{xy}^{(\rm el)}(\mathbf{\Gamma}) + 2\Delta\tilde{\mathcal{R}}_{\rm SS}^{(1)}(\mathbf{\Gamma})\right]$. We note that (i) the steady-state temperature $T_{\rm SS} = \beta_{\rm SS}^{-1}$ appears in Eq. (12), (ii) the steady-state average, $\langle\cdots\rangle_{\rm SS} \equiv \int d\mathbf{\Gamma}\,\rho_{\rm SS}(\mathbf{\Gamma})\cdots$, is independent of the equilibrium temperature for $\rho_{\rm eq}(\mathbf{\Gamma})$, (iii) the problem reduces to an "equilibrium" one with an effective Hamiltonian $\mathcal{H}_{\rm eff}(\mathbf{\Gamma}) = T_{\rm SS} I_{\rm SS}(\mathbf{\Gamma})$ and the temperature $T_{\rm SS}$. Because the nonequilibrium contribution is small, we further expand $\rho_{\rm SS}(\mathbf{\Gamma})$ as

$$\rho_{\rm SS}(\mathbf{\Gamma}) \approx \frac{e^{-\beta_{\rm SS}^*\mathcal{H}^*(\mathbf{\Gamma})}\left[1+\tilde{\tau}_{\rm rel}\tilde{\Omega}_{\rm SS}(\mathbf{\Gamma})\right]}{\mathcal{Z}} \tag{13}$$

with $\mathcal{Z} \approx \int d\mathbf{\Gamma}\, e^{-\beta_{\rm SS}^*\mathcal{H}^*(\mathbf{\Gamma})}\left[1+\tilde{\tau}_{\rm rel}\tilde{\Omega}_{\rm SS}(\mathbf{\Gamma})\right]$. An approximate expression for $\langle A(\mathbf{\Gamma})\rangle_{\rm SS}$ is obtained as

$$\langle A(\mathbf{\Gamma})\rangle_{\rm SS} \approx \langle A(\mathbf{\Gamma})\rangle_{\rm eq} + \tilde{\tau}_{\rm rel}\left\langle A(\mathbf{\Gamma})\tilde{\Omega}_{\rm SS}(\mathbf{\Gamma})\right\rangle_{\rm eq}, \tag{14}$$

where $\langle\cdots\rangle_{\rm eq} = \int d\mathbf{\Gamma}\, e^{-\beta_{\rm SS}^*\mathcal{H}^*(\mathbf{\Gamma})}\cdots$ is the average with respect to the canonical distribution at $T_{\rm SS}$. It should be noted that Eq. (14) is the result of an exponential damping in the stress-stress correlation function in the Green-Kubo formula.

So far $T_{\rm SS}$ is undetermined. We attempt to determine $T_{\rm SS}$ by imposing the energy balance

$$\left\langle \dot{\mathcal{H}}(\mathbf{\Gamma})\right\rangle_{\rm SS} = -\dot{\gamma}V\langle\sigma_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} - 2\langle\mathcal{R}(\mathbf{\Gamma})\rangle_{\rm SS} = 0. \tag{15}$$

The explicit form of $T_{\rm SS}$ will be given in Eq. (16).

Shear viscosity and temperature.– Now we calculate the steady-state average of the shear stress and the energy dissipation rate by Eq. (14) and derive an explicit expression for $T_{\rm SS}$. First, $\langle\sigma_{xy}(\mathbf{\Gamma})\rangle_{\rm SS}$ is approximately given by $\langle\tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} \approx -\tilde{\tau}_{\rm rel}\dot{\tilde{\gamma}}\beta_{\rm SS}^* V^*\left\langle \tilde{\sigma}_{xy}^{(\rm el)}(\mathbf{\Gamma})\tilde{\sigma}_{xy}^{(\rm el)}(\mathbf{\Gamma})\right\rangle_{\rm eq}$. Similarly, the leading contribution gives $\left\langle\tilde{\mathcal{R}}(\mathbf{\Gamma})\right\rangle_{\rm SS} \approx \left\langle\tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})\right\rangle_{\rm eq} - 2\tilde{\tau}_{\rm rel}\beta_{\rm SS}^*\left\langle\tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})\Delta\tilde{\mathcal{R}}_{\rm SS}^{(1)}(\mathbf{\Gamma})\right\rangle_{\rm eq}$. Thus, we obtain the steady-state temperature from Eq. (D23) as

$$T_{\rm SS}^* = \frac{3\dot{\tilde{\gamma}}^2}{32\pi}\frac{S}{R}, \tag{16}$$

where S and R are given by $S = 1 + \mathscr{S}_2\, n^* g_0(\varphi) + \mathscr{S}_3\, n^{*2} g_0(\varphi)^2 + \mathscr{S}_4\, n^{*3} g_0(\varphi)^3$ and $R = n^* g_0(\varphi)\left[\mathscr{R}_2' + \mathscr{R}_3' n^* g_0(\varphi)\right]$, with $\mathscr{S}_2 = 2\pi/15$, $\mathscr{S}_3 = -\pi^2/20$, $\mathscr{S}_4 = 3\pi^3/160$, $\mathscr{R}_2' = -3/4$, and $\mathscr{R}_3' = 7\pi/16$ [60]. We adopt the interpolation formula for hard spheres valid in the range $\varphi_f < \varphi < \varphi_J$ ($\varphi_f = 0.49$, $\varphi_J = 0.639$), $g_0(\varphi) = g_{\rm CS}(\varphi_f)(\varphi_f - \varphi_J)/(\varphi - \varphi_J)$, where $g_{\rm CS}(\varphi) = (1-\varphi/2)/(1-\varphi)^3$ is the formula by Carnahan and Starling valid at $\varphi < \varphi_f$ [62]. Note that we adopt the Kirkwood approximation for many-body correlations [60]. We further obtain the expression for the shear stress,

$$\langle\tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} = -\frac{3\sqrt{6}}{64\pi^{3/2}}\dot{\tilde{\gamma}}^2\frac{S^{3/2}}{R^{1/2}g_0(\varphi)}. \tag{17}$$

In the vicinity of the jamming point φ_J, S and R can be approximated as $S \approx \mathscr{S}_4\, n^{*3} g_0(\varphi)^3$ and $R \approx \mathscr{R}_3'\, n^{*2} g_0(\varphi)^2$, respectively. This leads to the following expressions,

$$T_{\rm SS}^* \approx \frac{9\pi}{2240}\dot{\tilde{\gamma}}^2 n^* g_0(\varphi), \tag{18}$$

$$\langle\tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} \approx -\frac{27\pi^{5/2}}{10240\sqrt{35}}\dot{\tilde{\gamma}}^2 n^{*7/2} g_0(\varphi)^{5/2}, \tag{19}$$

from which we obtain

$$T_{\rm SS}^* \sim (\varphi_J - \varphi)^{-1}, \tag{20}$$

$$\langle\tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} \sim (\varphi_J - \varphi)^{-5/2}. \tag{21}$$

From Eq. (17), we obtain the shear viscosity $\eta^* = -\langle\tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS}/\dot{\tilde{\gamma}} \sim (\varphi_J - \varphi)^{-5/2}$, or for $\tilde{\eta}' = -\langle\tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS}/(\dot{\tilde{\gamma}}\sqrt{T_{\rm SS}^*}) \propto -\langle\tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS}/\dot{\tilde{\gamma}}^2$,

$$\tilde{\eta}' \sim (\varphi_J - \varphi)^{-2}. \tag{22}$$

These results in Eq. (20)–(22) are consistent with the previous observations [26–30].

Comparison with simulation.– In order to verify the validity of the theory, we compare the theory with the molecular dynamics (MD) simulation. The parameters in the MD are $N = 2000$, $\epsilon = 0.018375$, and $\dot{\gamma}^* = 10^{-3}$, 10^{-4}, 10^{-5}. This condition corresponds to $e = 0.96$.

The shear viscosity $\tilde{\eta}'$ and $T_{\rm SS}$ are shown in Fig. 1, together with the results of the MD. We also show the log-log plots near φ_J as a function of $\varphi_J - \varphi$ in the inset. From the figure, we see that the theory agrees with the result of MD simulation for $\varphi < \varphi_J$ quantitatively without introducing any fitting parameter. The agreement is refined as the shear rate is decreased, where the hard-core limit is realized asymptotically. The smeared divergence in the vicinity of the jamming point observed for finite $\dot{\gamma}^*$ is a well-known feature of the soft-core MD. We stress that the theory predicts the divergence of *both* the shear viscosity and the granular temperature as $\varphi \to \varphi_J$, in contrast to the kinetic theory of inelastic spheres, where the shear viscosity behaves as $\tilde{\eta}' \sim (\varphi_J - \varphi)^{-1}$ and $T_{\rm SS}$ remains finite [35]. On the other hand, in the MD, the shear viscosity behaves as $\tilde{\eta}' \sim (\varphi_J - \varphi)^{-2}$, in accordance with the theory. We note that the agreement between the theory and MD is relatively poor for $T_{\rm SS}$, though we have not clarified its reason.

The relaxation time, Eq. (7), is shown in Fig. 2, together with the result of the MD. The result of the MD is extracted from fitting by an exponential function for the

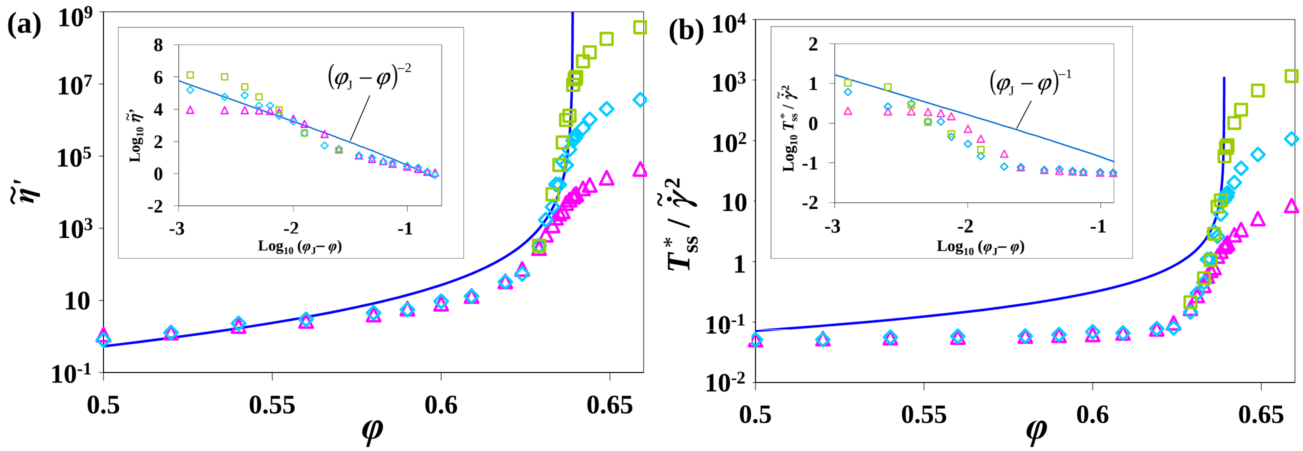

FIG. 1. (Color online) The density dependence of (a) the shear viscosity $\tilde{\eta}'$ and (b) the granular temperature. The result of the theory is shown in (blue) solid line, while that for the MD is shown in (red) triangles, (blue) diamonds, and (green) rectangles for $\dot{\gamma}^* = 10^{-3}, 10^{-4}, 10^{-5}$. (Inset) The log-log plots for the results near $\varphi_J = 0.639$.

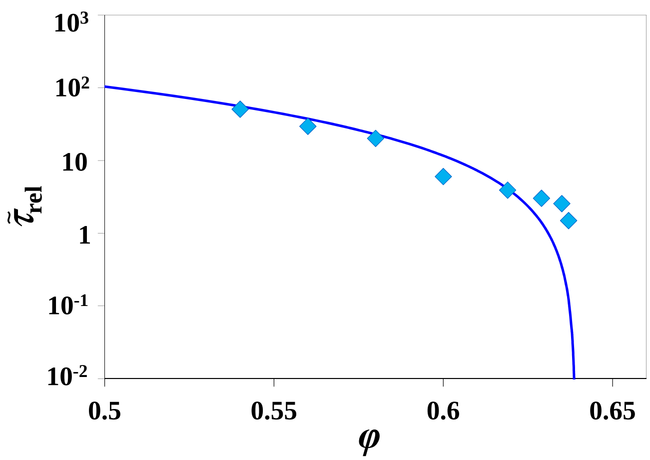

FIG. 2. (Color online) The density dependence of the relaxation time, $\tilde{\tau}_{\mathrm{rel}} = \epsilon\tau^*_{\mathrm{rel}}$. The result for the shear rate $\dot{\gamma}^* = 10^{-4}$ is shown in (blue) solid line for the theory and (blue) diamonds for the MD simulation.

transient data of the temperature relaxing to the steady-state. We see that Eq. (7) is quantitatively valid for $\varphi < 0.63$.

Discussions.– From Eqs. (16) and (17), we see that the theory is subjected to the Bagnold scaling. The result of MD shows that the discrepancy from the Bagnold scaling becomes significant for $\varphi > 0.635$. Hence, there is room for improving the theory to cover the non-Bagnold regime. From the phenomenological scaling of jammed granules, the viscosity exhibits $\eta \sim |\varphi_J - \varphi|^{y_\phi(1-2/y_\gamma)}$, where y_ϕ and y_γ are the scaling exponents for $\sigma_{xy} \sim (\varphi - \varphi_J)^{y_\phi}$ for $\varphi > \varphi_J$ and $\sigma_{xy} \sim \dot{\gamma}^{y_\gamma}$ at $\varphi = \varphi_J$ [18]. If we assume $y_\phi = 1$ as in Refs. [2, 3, 18, 19], we have $y_\gamma = 4/7$, which is close to the value of Ref. [17] [63] . For strongly dissipative situations, higher-order terms might alter the exponents of the divergences. Such a contribution will be discussed elsewhere.

Concluding Remarks.– We have developed a theory for jammed frictionless granular particles subjected to a uniform shear with the aid of an approximate nonequilibrium steady-state distribution function, and have shown that it remarkably agrees with the result of the MD simulation below the jamming point without introducing any fitting parameter. There are many future tasks for the application of our theory, such as the emergence of the shear modulus above the jamming point [2–5], the effect of friction for grains where the discontinuous shear thickening appears [64], the drag force acting on the pulling tracer [65–69], etc. Moreover, we should stress that the framework of our theory is quite generic. Indeed, we believe that the divergence of the viscosity for colloidal suspensions, $\eta \sim (\varphi_J - \varphi)^{-2}$ [70], can be understood by our framework. Therefore, the theory is expected to be applicable to a wide variety of phenomena in nonequilibrium processes.

The authors are grateful to S.-H. Chong and M. Otsuki for their contributions in the early stage of this work and extensive discussions. They also thank M. Fuchs, M. E. Cates, M. Wyart, K. Saitoh, A. Ikeda, and K. Kanazawa for fruitful discussions and comments, and M. Otsuki and S. Takada for providing the prototype of the program for the MD simulation. This work was partially supported by Grant-in-Aids for scientific research (Grant Nos. 25287098). The MD simulation for this work has been carried out at the computer facilities at the Yukawa Institute for Theoretical Physics, Kyoto University.

Appendix A: Microscopic starting equations

We introduce the system we consider, i.e. a three-dimensional system of N smooth granular particles of mass m in a volume V subjected to stationary shearing characterized by the shear-rate tensor $\dot{\gamma}_{\mu\nu} = \dot{\gamma}\delta_{\mu x}\delta_{\nu y}$. The time evolution of the system is determined by the

Newton's equation of motion,

$$m\ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i^{(\mathrm{el})} + \boldsymbol{F}_i^{(\mathrm{vis})} \ \ (i = 1, \cdots, N), \tag{A1}$$

under a suitable boundary condition such as the Lees-Edwards boundary condition [58], accounting for the stationary shearing. Here, $\boldsymbol{r}_i$ refers to the position of the i-th particle, and the dot denotes the time derivative. $\boldsymbol{F}_i^{(\mathrm{el})} = \sum_j' \boldsymbol{F}_{ij}^{(\mathrm{el})}$ is the conservative force, and is given by a sum $\sum_j'$ of forces exerted on the i-th particle by other particles,

$$\boldsymbol{F}_{ij}^{(\mathrm{el})} = -\frac{\partial u(r_{ij})}{\partial \boldsymbol{r}_{ij}} = \Theta(d - r_{ij}) f(d - r_{ij}) \hat{\boldsymbol{r}}_{ij}. \tag{A2}$$

Here, d is the diameter of the particle; $u(r)$ is the pair potential; $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$, $r_{ij} = |\boldsymbol{r}_{ij}|$, $\hat{\boldsymbol{r}}_{ij} = \boldsymbol{r}_{ij}/r_{ij}$; and $\Theta(x)$ is the Heaviside's step function, which is 1 for $x > 0$ and 0 otherwise. Although a realistic elastic force might be Hertzian, where $f(x) \propto x^{3/2}$ for a three-dimensional system, we adopt the linear spring model, $f(x) = \kappa x$ ($\kappa > 0$), for simplicity. $\boldsymbol{F}_i^{(\mathrm{vis})} = \sum_j' \boldsymbol{F}_{ij}^{(\mathrm{vis})}$ denotes the viscous dissipative force which is represented by a sum of pairwise contact forces,

$$\boldsymbol{F}_{ij}^{(\mathrm{vis})} = -\zeta \Theta(d - r_{ij}) \hat{\boldsymbol{r}}_{ij} (\boldsymbol{v}_{ij} \cdot \hat{\boldsymbol{r}}_{ij}). \tag{A3}$$

Here, $\boldsymbol{v}_{ij} \equiv \boldsymbol{v}_i - \boldsymbol{v}_j$ with $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$ refers to the relative velocity of colliding particles, and ζ (> 0) is a viscous constant corresponding to the harmonic potential. This model for sheared granular systems has been widely adopted in computer-simulation studies. Notice that the equation of motion is not invariant under the time-reversal map, since $\boldsymbol{F}_{ij}^{(\mathrm{vis})}$ changes sign for $\{\boldsymbol{r}_i, \boldsymbol{v}_i\} \to \{\boldsymbol{r}_i, -\boldsymbol{v}_i\}$.

Instead of Eq. (A1), we consider the following equation of motion valid for dense sheared granular systems,

$$\boldsymbol{p}_i = m\left(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{r}_i\right), \tag{A4a}$$
$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i^{(\mathrm{el})} + \boldsymbol{F}_i^{(\mathrm{vis})} - \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{p}_i, \tag{A4b}$$

which is called the Sllod equation [58]. Here, $\boldsymbol{p}_i$ is the fluctuation of momentum around the steady-shear momentum and referred to as the peculiar, or thermal, momentum. The use of the Sllod equation is restricted to uniform shear, which is in general not realized in realistic granular systems. However, it is a valid idealization in the high-density regime or for the flow on an inclined plane [40, 71, 72], where the profile of the shear velocity is well approximated as linear except for the region near the boundary. Hence, we adopt the Sllod equation to address the rheology of dense sheared granular liquids.

The internal energy of the system is given by

$$\mathcal{H}(\boldsymbol{\Gamma}) = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m} + {\sum_{i,j}}' u(r_{ij}), \tag{A5}$$

where $\boldsymbol{\Gamma} = \{\boldsymbol{r}_i, \boldsymbol{p}_i\}_{i=1}^N$ is the phase-space coordinate and ${\sum_{i,j}}'$ is the summation under the condition $i \neq j$. The rate of change of the internal energy $\dot{\mathcal{H}}(\boldsymbol{\Gamma})$ can be calculated, together with the Sllod equation, Eq. (A4), as

$$\begin{aligned}
\dot{\mathcal{H}} &= \sum_{i=1}^{N} \frac{\boldsymbol{p}_i}{m} \cdot \dot{\boldsymbol{p}}_i + {\sum_{i,j}}' \frac{\partial u(r_{ij})}{\partial \boldsymbol{r}_i} \cdot \dot{\boldsymbol{r}}_i \\
&= \sum_{i=1}^{N} \frac{\boldsymbol{p}_i}{m} \cdot \left[\boldsymbol{F}_i^{(\mathrm{el})} + \boldsymbol{F}_i^{(\mathrm{vis})} - \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{p}_i\right] - \sum_{i=1}^{N} \boldsymbol{F}_i^{(\mathrm{el})} \cdot \left[\frac{\boldsymbol{p}_i}{m} + \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{r}_i\right] \\
&= -\dot{\gamma} \sum_{i=1}^{N} \left[\frac{p_{i,x} p_{i,y}}{m} + y_i F_{i,x}^{(\mathrm{el})}\right] + \sum_{i=1}^{N} \frac{\boldsymbol{p}_i}{m} \cdot \boldsymbol{F}_i^{(\mathrm{vis})} \\
&= -\dot{\gamma} \sum_{i=1}^{N} \left[\frac{p_{i,x} p_{i,y}}{m} + y_i \left(F_{i,x}^{(\mathrm{el})} + F_{i,x}^{(\mathrm{vis})}\right)\right] + \sum_{i=1}^{N} \dot{\boldsymbol{r}}_i \cdot \boldsymbol{F}_i^{(\mathrm{vis})} \\
&= -\dot{\gamma} V \sigma_{xy} - 2\mathcal{R}.
\end{aligned} \tag{A6}$$

Here, $\sigma_{xy}(\boldsymbol{\Gamma})$ is the xy-component of the stress tensor,

$$\sigma_{\mu\nu}(\boldsymbol{\Gamma}) = \frac{1}{V} \sum_{i=1}^{N} \left[\frac{p_{i,\mu} p_{i,\nu}}{m} + r_{i,\nu} \left(F_{i,\mu}^{(\mathrm{el})} + F_{i,\mu}^{(\mathrm{vis})}\right)\right] \tag{A7}$$

and

$$\begin{aligned}
\mathcal{R}(\boldsymbol{\Gamma}) &= -\frac{1}{2} \sum_{i=1}^{N} \dot{\boldsymbol{r}}_i \cdot \boldsymbol{F}_i^{(\mathrm{vis})} = -\frac{1}{4} {\sum_{i,j}}' \boldsymbol{v}_{ij} \cdot \boldsymbol{F}_{ij}^{(\mathrm{vis})} \\
&= \frac{\zeta}{4} {\sum_{i,j}}' \Theta(d - r_{ij}) (\boldsymbol{v}_{ij} \cdot \hat{\boldsymbol{r}}_{ij})^2
\end{aligned} \tag{A8}$$

corresponds to the Rayleigh's dissipation function [59]. Note that we have utilized $\boldsymbol{F}_i^{(\mathrm{vis})} = \sum_{j \neq i} \boldsymbol{F}_{ij}^{(\mathrm{vis})}$ and $\boldsymbol{F}_{ij}^{(\mathrm{vis})} = -\boldsymbol{F}_{ji}^{(\mathrm{vis})}$ in the last equality. The first term on the right hand side of Eq. (A6) is the work rate of shear exerted on the system and the second term is the energy dissipation rate due to inelastic collisions. In the steady state, the balance between these two are realized, i.e.

$$\dot{\mathcal{H}}(\boldsymbol{\Gamma}) = -\dot{\gamma} V \sigma_{xy}(\boldsymbol{\Gamma}) - 2\mathcal{R}(\boldsymbol{\Gamma}) = 0. \tag{A9}$$

Appendix B: Liouville equation

The equation of motion, i.e. the Sllod equation Eq. (A4), can be converted into the Liouville equation [58]. For hard-sphere granular materials, it is conventional to adopt the pseudo-Liouville equation [73–75]. However, we start with the Liouville equation for soft spheres and take the hard-core limit later, since there is a subtlety in expressing the shear stress and the dissipation function via the pseudo-Liouvillian. We present the Liouville equation for uniformly sheared granular systems in the following. It closely follows the formulation of Ref. [58], where similar explicit expressions can also be found in Refs. [76, 77].

The time evolution of phase variables, which do not explicitly depend on time and whose time dependence

comes solely from that of the phase-space point $\mathbf{\Gamma} = \{\boldsymbol{r}_i, \boldsymbol{p}_i\}_{i=1}^N$, is determined by

$$\frac{d}{dt}A(\mathbf{\Gamma}(t)) = \dot{\mathbf{\Gamma}} \cdot \frac{\partial}{\partial \mathbf{\Gamma}} A(\mathbf{\Gamma}(t)) \equiv i\mathcal{L}A(\mathbf{\Gamma}(t)). \tag{B1}$$

The operator $i\mathcal{L}$ is referred to as the Liouvillian. For Eq. (A4), we have

$$i\mathcal{L} = i\mathcal{L}^{(\mathrm{el})} + i\mathcal{L}_{\dot{\gamma}} + i\mathcal{L}^{(\mathrm{vis})}, \tag{B2}$$

where the elastic part ($i\mathcal{L}^{(\mathrm{el})}$), the shear part ($i\mathcal{L}_{\dot{\gamma}}$), and the viscous part ($i\mathcal{L}^{(\mathrm{vis})}$) are, respectively, given by

$$i\mathcal{L}^{(\mathrm{el})} = \sum_i \Big[\frac{\boldsymbol{p}_i}{m} \cdot \frac{\partial}{\partial \boldsymbol{r}_i} + \boldsymbol{F}_i^{(\mathrm{el})} \cdot \frac{\partial}{\partial \boldsymbol{p}_i} \Big], \tag{B3}$$

$$i\mathcal{L}_{\dot{\gamma}} = \sum_i \Big[(\dot{\boldsymbol{\gamma}} \cdot \boldsymbol{r}_i) \cdot \frac{\partial}{\partial \boldsymbol{r}_i} - (\dot{\boldsymbol{\gamma}} \cdot \boldsymbol{p}_i) \cdot \frac{\partial}{\partial \boldsymbol{p}_i} \Big], \tag{B4}$$

$$i\mathcal{L}^{(\mathrm{vis})} = \sum_i \boldsymbol{F}_i^{(\mathrm{vis})} \cdot \frac{\partial}{\partial \boldsymbol{p}_i}. \tag{B5}$$

The formal solution to Eq. (B1) can be written in terms of the propagator $\exp(i\mathcal{L}t)$ as

$$A(\mathbf{\Gamma}(t)) = \exp(i\mathcal{L}t) A(\mathbf{\Gamma}). \tag{B6}$$

Hereafter, the absence of the argument t implies that associated quantities are evaluated at $t = 0$, e.g. $\mathbf{\Gamma} = \mathbf{\Gamma}(0)$.

The Liouville equation for the nonequilibrium phase-space distribution function $\rho(\mathbf{\Gamma}, t)$ is given by

$$\frac{\partial \rho(\mathbf{\Gamma}, t)}{\partial t} = -\frac{\partial}{\partial \mathbf{\Gamma}} \cdot \Big[\dot{\mathbf{\Gamma}} \rho(\mathbf{\Gamma}, t) \Big] = -\Big[\dot{\mathbf{\Gamma}} \cdot \frac{\partial}{\partial \mathbf{\Gamma}} + \Lambda(\mathbf{\Gamma}) \Big] \rho(\mathbf{\Gamma}, t) \equiv -i\mathcal{L}^\dagger \rho(\mathbf{\Gamma}, t), \tag{B7}$$

where $i\mathcal{L}^\dagger(\mathbf{\Gamma}) = i\mathcal{L}(\mathbf{\Gamma}) + \Lambda(\mathbf{\Gamma})$ is referred to as the adjoint Liouvillian. Here, $\Lambda(\mathbf{\Gamma})$ denotes the phase-space contraction factor [58] which is defined by

$$\Lambda(\mathbf{\Gamma}) \equiv \frac{\partial}{\partial \mathbf{\Gamma}} \cdot \dot{\mathbf{\Gamma}} = \sum_i \Big(\frac{\partial}{\partial \boldsymbol{r}_i} \cdot \dot{\boldsymbol{r}}_i + \frac{\partial}{\partial \boldsymbol{p}_i} \cdot \dot{\boldsymbol{p}}_i \Big). \tag{B8}$$

For the Sllod equation Eq. (A4), one finds

$$\Lambda(\mathbf{\Gamma}) = -\frac{\zeta}{m} \sum_{i,j}{}' \, \Theta(d - r_{ij}) < 0. \tag{B9}$$

The formal solution of the Liouville equation (B7) reads

$$\rho(\mathbf{\Gamma}, t) = \exp(-i\mathcal{L}^\dagger t)\, \rho(\mathbf{\Gamma}, 0). \tag{B10}$$

Appendix C: The perturbation expansion of the Liouville equation

We attempt to derive the eigenfrequencies of the distribution function for dense sheared granular liquids by means of a perturbation expansion of Eq. (B7). In particular, we attempt to construct a Rayleigh-Schrödinger perturbation theory, where the dissipation and the shear are treated as perturbations.

In Eq. (B7), we consider the Laplace transform of $\rho(\mathbf{\Gamma}, t)$, i.e.

$$\Psi_n(\mathbf{\Gamma}) = \int_{-\infty}^0 dt \, e^{-z_n t} \rho(\mathbf{\Gamma}, t), \tag{C1}$$

which satisfies

$$i\mathcal{L}^\dagger(\mathbf{\Gamma}) \Psi_n(\mathbf{\Gamma}) = -z_n \Psi_n(\mathbf{\Gamma}). \tag{C2}$$

Here, n is an index for the Laplace modes, which is continuous or discrete. This is an eigenvalue equation for the adjoint Liouvillian. The distribution function is given by

$$\rho(\mathbf{\Gamma}, t) = \sum_{n=0}^\infty e^{z_n t} \Psi_n(\mathbf{\Gamma}), \tag{C3}$$

where the summation over n is an integral for continuous modes.

In order to perform a perturbation expansion, we first non-dimensionalize the observables by choosing the unit of mass, length, and time as m, d, and $\sqrt{m/\kappa}$, and introduce an infinitesimal parameter

$$\epsilon = \frac{\zeta}{\sqrt{\kappa m}} \ll 1. \tag{C4}$$

Note that the restitution coefficient e is related to ϵ as $e = \exp[-\zeta t_c/m]$ via $\zeta t_c/m = \pi\, \epsilon/\sqrt{2(1-\epsilon^2)}$, where $t_c = \pi/\sqrt{2\kappa/m - (\zeta/m)^2}$ is the duration of contact of the spheres [31]. For $\epsilon \ll 1$, the normalized energy dissipation rate $1-e^2$ can be approximated as $1-e^2 \approx 2\zeta t_c/m = \sqrt{2}\pi\epsilon + \mathcal{O}(\epsilon^3) \approx \sqrt{2}\pi\epsilon$. Together with $1-e^2 \approx 2(1-e)$, we have $\epsilon \approx \sqrt{2}(1-e)/\pi$ for $e \approx 1$. We attach a star $*$ to the non-dimensionalized quantites, e.g. $t^* = t\sqrt{\kappa/m}$.

Furthermore, we perform a scaling which leaves the steady-state granular temperature T_{SS}, which is defined by $\int d\mathbf{\Gamma}\, \rho_{\mathrm{SS}}(\mathbf{\Gamma}) \sum_{i=1}^N \boldsymbol{p}_i^2/(3Nm)$, where $\rho_{\mathrm{SS}}(\mathbf{\Gamma})$ is the steady-state value of $\rho(\mathbf{\Gamma}, t)$, to be independent of ϵ. This is because we are interested in the situation where the granular fluid keeps a meaningful fluid motion in the limit $\epsilon \to 0$. From dimensional analysis, $\langle \sigma_{xy}(\mathbf{\Gamma}) \rangle_{\mathrm{SS}}$ and $\langle \mathcal{R}(\mathbf{\Gamma}) \rangle_{\mathrm{SS}}$ scale as $\langle \sigma_{xy}(\mathbf{\Gamma}) \rangle_{\mathrm{SS}} \sim \dot{\gamma}\sqrt{mT_{\mathrm{SS}}}/d^2$ and $\langle \mathcal{R}(\mathbf{\Gamma}) \rangle_{\mathrm{SS}} \sim \zeta T_{\mathrm{SS}}/m$. From the energy balance, Eq. (A9), T_{SS} scales as $T_{\mathrm{SS}} \sim m^3 d^2 \dot{\gamma}^4/\zeta^2$, which leads to $T^*_{\mathrm{SS}} \sim \epsilon^{-2}\, \dot{\gamma}^{*4}$. Thus, if we require T^*_{SS} to be independent of ϵ, $\dot{\gamma}^*$ should scale as

$$\dot{\gamma}^* \sim \epsilon^{1/2}. \tag{C5}$$

To be specific, we introduce a scaled shear rate $\tilde{\dot{\gamma}}$ as

$$\dot{\gamma}^* = \epsilon^{1/2} \tilde{\dot{\gamma}}, \tag{C6}$$

where $\tilde{\dot{\gamma}}$ is independent of ϵ. We attach a tilde to the scaled quantities. Although implicit, it should be kept in

mind that the anisotropic shear stress $\sigma_{xy}(\boldsymbol{\Gamma})$ is proportional to the shear rate. This implies that $\sigma_{xy}(\boldsymbol{\Gamma})$ also scales as $\epsilon^{1/2}$. To illuminate this feature, we introduce the following scaling for the anisotropic quantities,

$$\boldsymbol{X}^T \cdot \boldsymbol{\Sigma}_{xy} \cdot \boldsymbol{X} = \epsilon^{1/2} \tilde{\boldsymbol{X}}^T \cdot \boldsymbol{\Sigma}_{xy} \cdot \tilde{\boldsymbol{X}}, \tag{C7}$$

where $\boldsymbol{X}$ is an arbitrary anisotropic vector and $(\boldsymbol{\Sigma}_{xy})_{\mu\nu} = \delta_{\mu x}\delta_{\nu y}$. For instance, we have $p^*_{i,x}p^*_{i,y} = \epsilon^{1/2}\tilde{p}_{i,x}\tilde{p}_{i,y}$.

Then, we can expand $i\mathcal{L}^\dagger(\boldsymbol{\Gamma})$ in terms of ϵ as

$$i\mathcal{L}^{*\dagger}(\boldsymbol{\Gamma}) = i\mathcal{L}^{(\mathrm{eq})*}(\boldsymbol{\Gamma}) + \epsilon i\tilde{\mathcal{L}}_1(\boldsymbol{\Gamma}), \tag{C8}$$

where the unperturbed operator $i\mathcal{L}^{(\mathrm{eq})*}(\boldsymbol{\Gamma})$ is given by Eq. (B3) and the perturbed operator $i\tilde{\mathcal{L}}_1(\boldsymbol{\Gamma})$ reads

$$i\tilde{\mathcal{L}}_1(\boldsymbol{\Gamma}) = i\tilde{\mathcal{L}}_{\dot\gamma}(\boldsymbol{\Gamma}) + i\tilde{\mathcal{L}}^{(\mathrm{vis})}(\boldsymbol{\Gamma}) + \tilde{\Lambda}(\boldsymbol{\Gamma}) \tag{C9}$$

with

$$i\tilde{\mathcal{L}}_{\dot\gamma}(\boldsymbol{\Gamma}) = \tilde{\dot\gamma}\sum_{i=1}^N \left[\tilde{y}_i \frac{\partial}{\partial \tilde{x}_i} - \tilde{p}_{i,y}\frac{\partial}{\partial \tilde{p}_{i,x}}\right], \tag{C10}$$

$$i\tilde{\mathcal{L}}^{(\mathrm{vis})}(\boldsymbol{\Gamma}) = \sum_{i=1}^N \tilde{\boldsymbol{F}}_i^{(\mathrm{vis})} \cdot \frac{\partial}{\partial \boldsymbol{p}_i^*}, \tag{C11}$$

$$\tilde{\boldsymbol{F}}_i^{(\mathrm{vis})} = -\sum_{j\neq i} \Theta(1-r^*_{ij})(\dot{\boldsymbol{r}}^*_{ij}\cdot\hat{\boldsymbol{r}}_{ij})\hat{\boldsymbol{r}}_{ij}, \tag{C12}$$

$$\tilde{\Lambda}(\boldsymbol{\Gamma}) = -\sum_{i,j}{}^{\prime}\,\Theta(1-r^*_{ij}), \tag{C13}$$

respectively. Accordingly, we expand the distribution function and the eigenvalue as

$$\Psi^*_n(\boldsymbol{\Gamma}) = \rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})\left[\Psi_n^{(0)*}(\boldsymbol{\Gamma}) + \epsilon\tilde{\Psi}_n^{(1)}(\boldsymbol{\Gamma})\right] + \mathcal{O}(\epsilon^2), \tag{C14}$$

$$z^*_n = z_n^{(0)*} + \epsilon\tilde{z}_n^{(1)} + \mathcal{O}(\epsilon^2), \tag{C15}$$

where $\rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})$ is the canonical distribution. Note that $\rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})$ satisfies

$$i\mathcal{L}^{(\mathrm{eq})*}(\boldsymbol{\Gamma})\rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma}) = 0 \tag{C16}$$

by virtue of the conservation of the internal energy. From Eqs. (C2), (C8), (C14), and (C15), we obtain

$$i\mathcal{L}^{(\mathrm{eq})*}(\boldsymbol{\Gamma})\Psi_n^{(0)*}(\boldsymbol{\Gamma}) = -z_n^{(0)*}\Psi_n^{(0)*}(\boldsymbol{\Gamma}), \tag{C17}$$

$$\begin{aligned} &i\mathcal{L}^{(\mathrm{eq})*}(\boldsymbol{\Gamma})\tilde{\Psi}_n^{(1)}(\boldsymbol{\Gamma}) + \rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})^{-1} i\tilde{\mathcal{L}}_1(\boldsymbol{\Gamma})\rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})\Psi_n^{(0)*}(\boldsymbol{\Gamma}) \\ &\quad = -z_n^{(0)*}\tilde{\Psi}_n^{(1)}(\boldsymbol{\Gamma}) - \tilde{z}_n^{(1)}\Psi_n^{(0)*}(\boldsymbol{\Gamma}). \end{aligned} \tag{C18}$$

1. Eigenequation for the zero modes

The unperturbed operator $i\mathcal{L}^{(\mathrm{eq})*}(\boldsymbol{\Gamma})$ has degenerate five zero-modes, which we denote by $\phi^*_\alpha(\boldsymbol{\Gamma})$ ($\alpha = 1, \cdots, 5$),

$$i\mathcal{L}^{(\mathrm{eq})*}(\boldsymbol{\Gamma})\phi^*_\alpha(\boldsymbol{\Gamma}) = 0 \quad (\alpha = 1, \cdots, 5). \tag{C19}$$

Explicitly, they are given by

$$\phi^*_\alpha(\boldsymbol{\Gamma}) \propto \left\{1,\, \sum_{i=1}^N p^*_{i,x},\, \sum_{i=1}^N p^*_{i,y},\, \sum_{i=1}^N p^*_{i,z},\, \mathcal{H}^*(\boldsymbol{\Gamma})\right\}. \tag{C20}$$

The equalities

$$i\mathcal{L}^{(\mathrm{eq})*}(\boldsymbol{\Gamma})\sum_{i=1}^N p^*_{i,\mu} = \sum_{i=1}^N F^{(\mathrm{el})*}_{i,\mu} = 0 \ (\mu = x, y, z) \tag{C21}$$

follow from the conservation of the momentum. We choose $\{\phi^*_\alpha(\boldsymbol{\Gamma})\}$ ($\alpha = 1, \cdots, 5$) to be orthogonal, i.e.

$$\int d\boldsymbol{\Gamma}^*\, \rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})\phi^*_\alpha(\boldsymbol{\Gamma})\phi^*_{\alpha'}(\boldsymbol{\Gamma}) \propto \delta_{\alpha\alpha'}. \tag{C22}$$

Specifically, we adopt

$$\phi^*_1(\boldsymbol{\Gamma}) = 1, \tag{C23}$$

$$\phi^*_2(\boldsymbol{\Gamma}) = \frac{1}{\sqrt{\frac{3}{2}NT^*}}\left(\sum_{i=1}^N \frac{\boldsymbol{p}_i^{*2}}{2} - \frac{3}{2}NT^*\right), \tag{C24}$$

$$\phi^*_\alpha(\boldsymbol{\Gamma}) = \frac{1}{\sqrt{NT^*}}\sum_{i=1}^N p^*_{i,\lambda} \tag{C25}$$

where $\lambda = x$, y, and z correspond to $\alpha = 3$, 4, and 5, respectively. This leads to

$$\int d\boldsymbol{\Gamma}^*\, \rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})\phi^*_\alpha(\boldsymbol{\Gamma})\phi^*_{\alpha'}(\boldsymbol{\Gamma}) = \delta_{\alpha\alpha'}. \tag{C26}$$

As for the energy eigenmode, $\phi^*_2(\boldsymbol{\Gamma})$, we consider only the kinetic energy. We expect this treatment to be valid for the unperturbed zero-modes in the hard-core limit.

In the following, we work in the 5-dimensional space spanned by Eqs. (C23)–(C25). Then, the distribution function is explicitly given by

$$\begin{aligned} \rho^*(\boldsymbol{\Gamma}, t) = &\sum_{\alpha=1}^5 e^{\epsilon\tilde{z}_\alpha^{(1)}t^* + \cdots}\rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma}) \\ &\times\left[\Psi_\alpha^{(0)*}(\boldsymbol{\Gamma}) + \epsilon\tilde{\Psi}_\alpha^{(1)}(\boldsymbol{\Gamma}) + \cdots\right]. \end{aligned} \tag{C27}$$

Since $\{\phi^*_\alpha(\boldsymbol{\Gamma})\}$ ($\alpha = 1, \cdots, 5$) are five-fold degenerate, we must choose an appropriate linear combination to construct the unperturbed distribution function,

$$\Psi_\alpha^{(0)*}(\boldsymbol{\Gamma}) = \sum_{\alpha'=1}^5 c_{\alpha\alpha'}\phi^*_{\alpha'}(\boldsymbol{\Gamma}), \tag{C28}$$

where $\{c_{\alpha\beta}\}_{\alpha,\beta=1}^5$ are the coefficients.

We next consider Eq. (C18). Since $z_\alpha^{(0)*} = 0$, we obtain

$$\begin{aligned} &\rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})i\mathcal{L}^{(\mathrm{eq})*}(\boldsymbol{\Gamma})\tilde{\Psi}_\alpha^{(1)}(\boldsymbol{\Gamma}) + i\tilde{\mathcal{L}}_1(\boldsymbol{\Gamma})\rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})\Psi_\alpha^{(0)*}(\boldsymbol{\Gamma}) \\ &\quad = -\tilde{z}_\alpha^{(1)}\rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})\Psi_\alpha^{(0)*}(\boldsymbol{\Gamma}). \end{aligned} \tag{C29}$$

By multiplying Eq. (C29) by $\phi^*_{\alpha'}(\boldsymbol{\Gamma})$ and integrating with respect to $\boldsymbol{\Gamma}^*$, we obtain

$$\begin{aligned} &\int d\boldsymbol{\Gamma}^*\, \phi^*_{\alpha'}(\boldsymbol{\Gamma})\left[i\tilde{\mathcal{L}}_1(\boldsymbol{\Gamma})\rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})\Psi_\alpha^{(0)*}(\boldsymbol{\Gamma})\right] \\ &\quad = -\tilde{z}_\alpha^{(1)}\int d\boldsymbol{\Gamma}^*\, \phi^*_{\alpha'}(\boldsymbol{\Gamma})\left[\rho^*_{\mathrm{eq}}(\boldsymbol{\Gamma})\Psi_\alpha^{(0)*}(\boldsymbol{\Gamma})\right], \end{aligned} \tag{C30}$$

which follows from the fact that the first term on the left-hand side of Eq. (C29) vanishes,

$$\int d\mathbf{\Gamma}^*\, \phi^*_{\alpha'}(\mathbf{\Gamma}) \left[\rho^*_{\rm eq}(\mathbf{\Gamma}) i\mathcal{L}^{({\rm eq})*}(\mathbf{\Gamma}) \tilde{\Psi}^{(1)}_\alpha(\mathbf{\Gamma})\right] = 0, \quad (C31)$$

due to $i\mathcal{L}^{({\rm eq})*}\phi^*_\alpha(\mathbf{\Gamma}) = 0$. From Eq. (C28), Eq. (C30) is expressed as

$$-\tilde{z}^{(1)}_\alpha c_{\alpha\alpha'} = \sum_{\alpha''} c_{\alpha\alpha''} W_{\alpha'\alpha''}, \quad (C32)$$

where

$$W_{\alpha'\alpha''} \equiv \int d\mathbf{\Gamma}^*\, \phi^*_{\alpha'}(\mathbf{\Gamma}) i\tilde{\mathcal{L}}_1(\mathbf{\Gamma}) \rho^*_{\rm eq}(\mathbf{\Gamma}) \phi^*_{\alpha''}(\mathbf{\Gamma}). \quad (C33)$$

We obtain five eigenvalue equations for $\tilde{z}^{(1)}_\alpha$ ($\alpha = 1, \cdots, 5$) from Eq. (C32),

$$\boldsymbol{W} \boldsymbol{c}^T_\alpha = -\tilde{z}^{(1)}_\alpha \boldsymbol{c}^T_\alpha, \quad (C34)$$

where $\boldsymbol{c}^T_\alpha$ is a vector which constitutes a matrix $\boldsymbol{c}^T$, i.e. $\boldsymbol{c}^T = \{\boldsymbol{c}^T_1, \cdots, \boldsymbol{c}^T_5\}$. Here, the superscript T denotes a transpose. Then, the eigen equations read

$$\det\left[\boldsymbol{W} + \tilde{z}^{(1)}_\alpha \mathbf{1}\right] = 0 \quad (\alpha = 1, \cdots, 5), \quad (C35)$$

where $\mathbf{1}$ is the identity matrix.

Now we calculate the matrix elements of $\boldsymbol{W}$, which can be decomposed into

$$\boldsymbol{W} = \boldsymbol{W}^{(\dot\gamma)} + \boldsymbol{W}^{({\rm vis})} + \boldsymbol{W}^{(\Lambda)}, \quad (C36)$$

where

$$W^{(\dot\gamma)}_{\alpha\alpha'} = \int d\mathbf{\Gamma}^*\, \phi^*_\alpha(\mathbf{\Gamma}) i\tilde{\mathcal{L}}_{\dot\gamma}(\mathbf{\Gamma}) \rho^*_{\rm eq}(\mathbf{\Gamma}) \phi^*_{\alpha'}(\mathbf{\Gamma}), \quad (C37)$$

$$W^{({\rm vis})}_{\alpha\alpha'} = \int d\mathbf{\Gamma}^*\, \phi^*_\alpha(\mathbf{\Gamma}) i\tilde{\mathcal{L}}^{({\rm vis})}(\mathbf{\Gamma}) \rho^*_{\rm eq}(\mathbf{\Gamma}) \phi^*_{\alpha'}(\mathbf{\Gamma}), \quad (C38)$$

$$W^{(\Lambda)}_{\alpha\alpha'} = \int d\mathbf{\Gamma}^*\, \phi^*_\alpha(\mathbf{\Gamma}) \tilde{\Lambda}(\mathbf{\Gamma}) \rho^*_{\rm eq}(\mathbf{\Gamma}) \phi^*_{\alpha'}(\mathbf{\Gamma}). \quad (C39)$$

For Eqs. (C37) and (C38), it is necessary to evaluate the integrands,

$$i\tilde{\mathcal{L}}_{\dot\gamma}(\mathbf{\Gamma}) \rho^*_{\rm eq}(\mathbf{\Gamma}) \phi^*_\alpha(\mathbf{\Gamma}) = \left[i\tilde{\mathcal{L}}_{\dot\gamma}(\mathbf{\Gamma}) \rho^*_{\rm eq}(\mathbf{\Gamma})\right] \phi^*_\alpha(\mathbf{\Gamma}) + \rho^*_{\rm eq}(\mathbf{\Gamma}) i\tilde{\mathcal{L}}_{\dot\gamma}(\mathbf{\Gamma}) \phi^*_\alpha(\mathbf{\Gamma}), \quad (C40)$$

$$i\tilde{\mathcal{L}}^{({\rm vis})}(\mathbf{\Gamma}) \rho^*_{\rm eq}(\mathbf{\Gamma}) \phi^*_\alpha(\mathbf{\Gamma}) = \left[i\tilde{\mathcal{L}}^{({\rm vis})}(\mathbf{\Gamma}) \rho^*_{\rm eq}(\mathbf{\Gamma})\right] \phi^*_\alpha(\mathbf{\Gamma}) + \rho^*_{\rm eq}(\mathbf{\Gamma}) i\tilde{\mathcal{L}}^{({\rm vis})}(\mathbf{\Gamma}) \phi^*_\alpha(\mathbf{\Gamma}). \quad (C41)$$

The Liouvillians act on $\rho^*_{\rm eq}(\mathbf{\Gamma})$ as

$$\begin{aligned}
&\rho^*_{\rm eq}(\mathbf{\Gamma})^{-1} i\tilde{\mathcal{L}}_{\dot\gamma}(\mathbf{\Gamma}) \rho^*_{\rm eq}(\mathbf{\Gamma}) \\
&\quad = -\beta^* \tilde{\dot\gamma} \sum_{i=1}^N \left[\tilde{y}_i \frac{\partial}{\partial \tilde{x}_i} - \tilde{p}_{i,y} \frac{\partial}{\partial \tilde{p}_{i,x}}\right] \mathcal{H}^*(\mathbf{\Gamma}) \\
&\quad = \beta^* \tilde{\dot\gamma} \sum_{i=1}^N \left[\tilde{p}_{i,x}\tilde{p}_{i,y} + \tilde{y}_i \tilde{F}^{({\rm el})}_{i,x}\right] = \beta^* \tilde{\dot\gamma} V^* \tilde{\sigma}^{({\rm el})}_{xy}(\mathbf{\Gamma}), \quad (C42)
\end{aligned}$$

$$\begin{aligned}
&\rho^*_{\rm eq}(\mathbf{\Gamma})^{-1} i\tilde{\mathcal{L}}^{({\rm vis})}(\mathbf{\Gamma}) \rho^*_{\rm eq}(\mathbf{\Gamma}) \\
&\quad = -\beta^* \sum_{i=1}^N \tilde{\boldsymbol{F}}^{({\rm vis})}_i \cdot \frac{\partial}{\partial \boldsymbol{p}^*_i} \mathcal{H}^*(\mathbf{\Gamma}) = -\beta^* \sum_{i=1}^N \boldsymbol{p}^*_i \cdot \tilde{\boldsymbol{F}}^{({\rm vis})}_i \\
&\quad = \frac{1}{2}\beta^* {\sum_{i,j}}' \Theta(1 - r^*_{ij}) \left(\dot{\boldsymbol{r}}^*_{ij} \cdot \hat{\boldsymbol{r}}_{ij}\right) \left(\boldsymbol{p}^*_{ij} \cdot \hat{\boldsymbol{r}}_{ij}\right) \\
&\quad = \frac{1}{2}\beta^* {\sum_{i,j}}' \Theta(1 - r^*_{ij}) \left(\left[\boldsymbol{p}^*_{ij} + \tilde{\dot\gamma} \tilde{y}_i \boldsymbol{e}_x\right] \cdot \hat{\boldsymbol{r}}_{ij}\right) \left(\boldsymbol{p}^*_{ij} \cdot \hat{\boldsymbol{r}}_{ij}\right) \\
&\quad = 2\beta^* \tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma}) - \epsilon^{1/2} \beta^* \tilde{\dot\gamma} V^* \tilde{\sigma}^{({\rm vis})(1)}_{xy}(\mathbf{\Gamma}) \\
&\quad \approx 2\beta^* \tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma}), \quad (C43)
\end{aligned}$$

where the scaled quantities $\tilde{\sigma}^{({\rm el})}_{xy}(\mathbf{\Gamma})$, $\tilde{\sigma}^{({\rm vis})(1)}_{xy}(\mathbf{\Gamma})$, and $\tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})$ are given by

$$\tilde{\sigma}^{({\rm el})}_{xy}(\mathbf{\Gamma}) = \frac{1}{V^*} \sum_{i=1}^N \left[\tilde{p}_{i,x}\tilde{p}_{i,y} + \tilde{y}_i \tilde{F}^{({\rm el})}_{i,x}\right], \quad (C44)$$

$$\tilde{\sigma}^{({\rm vis})(1)}_{xy}(\mathbf{\Gamma}) = -\frac{1}{2V^*} {\sum_{i,j}}' \left(\boldsymbol{p}^*_{ij} \cdot \hat{\boldsymbol{r}}_{ij}\right) \tilde{x}_{ij} \tilde{y}_{ij} r^*_{ij} \Theta(1 - r^*_{ij}), \quad (C45)$$

$$\tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma}) = \frac{1}{4} {\sum_{i,j}}' \left(\boldsymbol{p}^*_{ij} \cdot \hat{\boldsymbol{r}}_{ij}\right)^2 \Theta(1 - r^*_{ij}). \quad (C46)$$

On the other hand, the Liouvillians act on $\phi^*_\alpha(\mathbf{\Gamma})$ as

$$i\tilde{\mathcal{L}}_{\dot\gamma}(\mathbf{\Gamma}) \phi^*_\alpha(\mathbf{\Gamma}) = 0 \quad (\alpha = 1, 4, 5), \quad (C47)$$

$$i\tilde{\mathcal{L}}_{\dot\gamma}(\mathbf{\Gamma}) \phi^*_2(\mathbf{\Gamma}) = -\frac{\tilde{\dot\gamma}}{\sqrt{\frac{3}{2}NT^*}} \sum_{i=1}^N \tilde{p}_{i,x}\tilde{p}_{i,y}, \quad (C48)$$

$$i\tilde{\mathcal{L}}_{\dot\gamma}(\mathbf{\Gamma}) \phi^*_3(\mathbf{\Gamma}) = -\frac{\tilde{\dot\gamma}}{\sqrt{NT^*}} \sum_{i=1}^N \tilde{p}_{i,y}, \quad (C49)$$

and

$$i\tilde{\mathcal{L}}^{({\rm vis})}(\mathbf{\Gamma}) \phi^*_1(\mathbf{\Gamma}) = 0, \quad (C50)$$

$$i\tilde{\mathcal{L}}^{({\rm vis})}(\mathbf{\Gamma}) \phi^*_2(\mathbf{\Gamma}) = \frac{1}{\sqrt{\frac{3}{2}NT^*}} \sum_{i=1}^N \boldsymbol{p}^*_i \cdot \tilde{\boldsymbol{F}}^{({\rm vis})}_i = \frac{-2\tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})}{\sqrt{\frac{3}{2}NT^*}}, \quad (C51)$$

$$i\tilde{\mathcal{L}}^{({\rm vis})}(\mathbf{\Gamma}) \phi^*_\alpha(\mathbf{\Gamma}) = \sum_{i=1}^N \frac{\tilde{F}^{({\rm vis})}_{i,\lambda}}{\sqrt{NT^*}} \quad (\alpha = 3, 4, 5), \quad (C52)$$

where $\lambda = x$, y, and z correspond to $\alpha = 3$, 4, and 5, respectively. Hence, Eqs. (C37) and (C38) are recasted

in the form

$$W^{(\dot{\gamma})}_{\alpha\alpha'} = \int d\mathbf{\Gamma}^* \, \rho^*_{\rm eq}(\mathbf{\Gamma})\phi^*_\alpha(\mathbf{\Gamma})\beta^*\tilde{\dot{\gamma}}V^*\tilde{\sigma}^{(\rm el)}_{xy}(\mathbf{\Gamma})\phi^*_{\alpha'}(\mathbf{\Gamma}) + \int d\mathbf{\Gamma}^* \, \rho^*_{\rm eq}(\mathbf{\Gamma})\phi^*_\alpha(\mathbf{\Gamma})i\tilde{\mathcal{L}}_{\dot{\gamma}}(\mathbf{\Gamma})\phi^*_{\alpha'}(\mathbf{\Gamma}), \tag{C53}$$

$$W^{(\rm vis)}_{\alpha\alpha'} = \int d\mathbf{\Gamma}^* \, \rho^*_{\rm eq}(\mathbf{\Gamma})\phi^*_\alpha(\mathbf{\Gamma})2\beta^*\tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})\phi^*_{\alpha'}(\mathbf{\Gamma}) + \int d\mathbf{\Gamma}^* \, \rho^*_{\rm eq}(\mathbf{\Gamma})\phi^*_\alpha(\mathbf{\Gamma})i\tilde{\mathcal{L}}^{(\rm vis)}(\mathbf{\Gamma})\phi^*_{\alpha'}(\mathbf{\Gamma}). \tag{C54}$$

Together with Eq. (C39), we obtain

$$W_{\alpha\alpha'} = \int d\mathbf{\Gamma}^* \, \rho^*_{\rm eq}(\mathbf{\Gamma})\phi^*_\alpha(\mathbf{\Gamma}) \Big[\beta^*\tilde{\dot{\gamma}}V^*\tilde{\sigma}^{(\rm el)}_{xy}(\mathbf{\Gamma}) + 2\beta^*\Delta\tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma}) + i\tilde{\mathcal{L}}_{\dot{\gamma}}(\mathbf{\Gamma}) + i\tilde{\mathcal{L}}^{(\rm vis)}(\mathbf{\Gamma})\Big] \phi^*_{\alpha'}(\mathbf{\Gamma}), \tag{C55}$$

where $\Delta\tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})$ is given by

$$\Delta\tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma}) = \tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma}) + \frac{T^*}{2}\tilde{\Lambda}(\mathbf{\Gamma}). \tag{C56}$$

We denote the three components of $\boldsymbol{W}$ as

$$\boldsymbol{W} = \boldsymbol{W}^{(\rm s)} + \boldsymbol{W}^{(\rm a1)} + \boldsymbol{W}^{(\rm a2)}, \tag{C57}$$

where

$$W^{(\rm s)}_{\alpha\alpha'} \equiv \int d\mathbf{\Gamma}^* \, \rho^*_{\rm eq}(\mathbf{\Gamma})\phi^*_\alpha(\mathbf{\Gamma}) \times\beta^* \Big[\tilde{\dot{\gamma}}V^*\tilde{\sigma}^{(\rm el)}_{xy} + 2\Delta\tilde{\mathcal{R}}^{(1)}\Big] \phi^*_{\alpha'}(\mathbf{\Gamma}) \tag{C58}$$

is the symmetric part and

$$W^{(\rm a1)}_{\alpha\alpha'} \equiv \int d\mathbf{\Gamma}^* \, \rho^*_{\rm eq}(\mathbf{\Gamma})\phi^*_\alpha(\mathbf{\Gamma})i\tilde{\mathcal{L}}_{\dot{\gamma}}(\mathbf{\Gamma})\phi^*_{\alpha'}(\mathbf{\Gamma}), \tag{C59}$$

$$W^{(\rm a2)}_{\alpha\alpha'} \equiv \int d\mathbf{\Gamma}^* \, \rho^*_{\rm eq}(\mathbf{\Gamma})\phi^*_\alpha(\mathbf{\Gamma})i\tilde{\mathcal{L}}^{(\rm vis)}(\mathbf{\Gamma})\phi^*_{\alpha'}(\mathbf{\Gamma}), \tag{C60}$$

are the asymmetric parts. The elements of $\boldsymbol{W}^{(\rm a1)}$ and $\boldsymbol{W}^{(\rm a2)}$ are given from Eqs. (C47)–(C49) and Eqs. (C50)–(C52) as

$$\boldsymbol{W}^{(\rm a1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{\dot{\gamma}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{C61}$$

$$\boldsymbol{W}^{(\rm a2)} \approx \begin{bmatrix} 0 & -\sqrt{\frac{2}{3}N}\mathscr{G} & 0 & 0 & 0 \\ 0 & -\frac{2}{3}\mathscr{G} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{C62}$$

where $\mathscr{G}$ is given by

$$\mathscr{G} = n^* \int d^3\boldsymbol{r}^* \, g(r^*, \varphi)\Theta(1 - r^*). \tag{C63}$$

Here, $g(r, \varphi)$ is the radial distribution function at equilibrium defined by $\langle {\sum_{i,j}}' \delta(\boldsymbol{r} - \boldsymbol{r}_i)\rangle_{\rm eq} = Nng(r, \varphi)$. In Eq. (C62), terms of order $\epsilon^{1/2}$ are neglected. The elements of $\boldsymbol{W}^{(s)}$ are given by

$$\boldsymbol{W}^{(\rm s)} = \begin{bmatrix} 0 & \sqrt{\frac{2}{3}N}\mathscr{G} & 0 & 0 & 0 \\ \sqrt{\frac{2}{3}N}\mathscr{G} & \frac{4}{3}\mathscr{G} & 0 & 0 & 0 \\ 0 & 0 & 2\mathscr{G}^{xx} & \tilde{\dot{\gamma}} + 2\mathscr{G}^{xy} & 2\mathscr{G}^{xz} \\ 0 & 0 & \tilde{\dot{\gamma}} + 2\mathscr{G}^{yx} & 2\mathscr{G}^{yy} & 2\mathscr{G}^{yz} \\ 0 & 0 & 2\mathscr{G}^{zx} & 2\mathscr{G}^{zy} & 2\mathscr{G}^{zz} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{\frac{2}{3}N}\mathscr{G} & 0 & 0 & 0 \\ \sqrt{\frac{2}{3}N}\mathscr{G} & \frac{4}{3}\mathscr{G} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3}\mathscr{G} & \tilde{\dot{\gamma}} & 0 \\ 0 & 0 & \tilde{\dot{\gamma}} & \frac{2}{3}\mathscr{G} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3}\mathscr{G} \end{bmatrix} \tag{C64}$$

with $\mathscr{G}^{\mu\nu} = n^* \int d^3\boldsymbol{r}^* \, g(r^*, \varphi)\Theta(1 - r^*)\hat{r}^\mu\hat{r}^\nu$, where the anisotropic components vanish, $\mathscr{G}^{\mu\nu} = 0$ ($\mu \neq \nu$), and the isotropic components are given by $\mathscr{G}^{xx} = \mathscr{G}^{yy} = \mathscr{G}^{zz} = \mathscr{G}/3$. From Eqs. (C57), (C61), (C62), and (C64), we obtain

$$\boldsymbol{W} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{2}{3}N}\mathscr{G} & \frac{2}{3}\mathscr{G} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3}\mathscr{G} & \tilde{\dot{\gamma}} & 0 \\ 0 & 0 & 0 & \frac{2}{3}\mathscr{G} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3}\mathscr{G} \end{bmatrix}. \tag{C65}$$

Then, from Eq. (C35), the eigenvalues $\tilde{z}^{(1)}_\alpha$ ($\alpha = 1, \cdots, 5$) are obtained as solutions of the following equation for λ,

$$
\det \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ \sqrt{\frac{2}{3}N\mathscr{G}} & \frac{2}{3}\mathscr{G}+\lambda & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3}\mathscr{G}+\lambda & \tilde{\dot{\gamma}} & 0 \\ 0 & 0 & 0 & \frac{2}{3}\mathscr{G}+\lambda & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3}\mathscr{G}+\lambda \end{bmatrix} = 0. \tag{C66}
$$

2. Eigenvalues of the first order $z_\alpha^{(1)}$

From Eq. (C66), the eigenvalues are given by

$$
\tilde{z}_1^{(1)} = 0, \tag{C67}
$$

$$
\tilde{z}_\alpha^{(1)} = -\frac{2}{3}\mathscr{G} \quad (\alpha = 2, 3, 4, 5), \tag{C68}
$$

from which we see that the four non-zero modes are degenerate and hence can be approximated as a single relaxation mode. The relaxation time for this mode is given by

$$
\tau_{\rm rel}^* \approx -\frac{1}{\epsilon \tilde{z}_\alpha^{(1)}} = \left[\frac{2}{3}\epsilon\,\mathscr{G}\right]^{-1}. \tag{C69}
$$

In the hard-core limit, $\mathscr{G}$ reduces to

$$
\mathscr{G} \to \sqrt{\pi}\,\omega_E^*(T^*), \tag{C70}
$$

(cf. Eq. (G11) which will be shown later), where $\omega_E(T) = 4\sqrt{\pi}\,n\sqrt{T/m}\,g_0(\varphi)d^2$ is the Enskog frequency of collisions, and $g_0(\varphi)$ is the first-peak value of the radial distribution function, i.e. $g_0(\varphi) = g(d,\varphi)$. From Eqs. (C69) and (C70), we obtain

$$
\tau_{\rm rel}^* = \left[\frac{2\sqrt{\pi}}{3}\epsilon\,\omega_E^*(T^*)\right]^{-1}. \tag{C71}
$$

Appendix D: Steady-state distribution function

We derive an approximate explicit expression for the nonequilibrium steady-state distribution function, with the aid of the relaxation time derived in Sec C. We start from an equilibrium state at $t \to -\infty$ and evolve the system with shear and dissipation, reaching a nonequilibrium steady state at $t = 0$. A formal but *exact* expression for the nonequilibrium steady-state distribution function is given by [58]

$$
\rho_{\rm SS}^{(\rm ex)}(\mathbf{\Gamma}) = \exp\left[\int_{-\infty}^0 d\tau\,\Omega_{\rm eq}(\mathbf{\Gamma}(-\tau))\right]\rho_{\rm eq}(\mathbf{\Gamma}(-\infty)), \tag{D1}
$$

which satisfies $i\mathcal{L}^\dagger\rho_{\rm SS}^{(\rm ex)}(\mathbf{\Gamma}) = 0$. Here,

$$
\begin{aligned}
\Omega_{\rm eq}(\mathbf{\Gamma}) &= \beta_{\rm eq}\dot{\mathcal{H}}(\mathbf{\Gamma}) - \Lambda(\mathbf{\Gamma}) \\
&= -\beta_{\rm eq}\left[\dot{\gamma}V\sigma_{xy}(\mathbf{\Gamma}) + 2\mathcal{R}(\mathbf{\Gamma})\right] - \Lambda(\mathbf{\Gamma})
\end{aligned} \tag{D2}
$$

is the work function for the equilibrium distribution $\rho_{\rm eq}(\mathbf{\Gamma}) = e^{-\beta_{\rm eq}\mathcal{H}(\mathbf{\Gamma})}/\int d\mathbf{\Gamma}\,e^{-\beta_{\rm eq}\mathcal{H}(\mathbf{\Gamma})}$ with the temperature $T_{\rm eq} = \beta_{\rm eq}^{-1}$. To proceed, it is convenient to cast $\Omega_{\rm eq}(\mathbf{\Gamma})$ in another form. Note that the shear-stress tensor $\sigma_{xy}(\mathbf{\Gamma})$ decomposes into

$$
\sigma_{xy}(\mathbf{\Gamma}) = \sigma_{xy}^{(\rm el)}(\mathbf{\Gamma}) + \sigma_{xy}^{(\rm vis)}(\mathbf{\Gamma}), \tag{D3}
$$

where

$$
\sigma_{xy}^{(\rm el)}(\mathbf{\Gamma}) \equiv \frac{1}{V}\sum_{i=1}^N\left[\frac{p_{i,x}p_{i,y}}{m} + y_iF_{i,x}^{(\rm el)}\right], \tag{D4}
$$

$$
\sigma_{xy}^{(\rm vis)}(\mathbf{\Gamma}) \equiv \frac{1}{V}\sum_{i=1}^N y_iF_{i,x}^{(\rm vis)}. \tag{D5}
$$

From $\boldsymbol{F}_i^{(\rm vis)} = \sum_j{}' \boldsymbol{F}_{ij}^{(\rm vis)} = -\sum_j{}' \boldsymbol{F}_{ji}^{(\rm vis)}$, $\sigma_{xy}^{(\rm vis)}(\mathbf{\Gamma})$ can be rewritten as

$$
\begin{aligned}
\sigma_{xy}^{(\rm vis)}(\mathbf{\Gamma}) &= \frac{1}{2V}\sum_{i,j}{}' y_{ij}F_{ij,x}^{(\rm vis)} \\
&= -\frac{\zeta}{2V}\sum_{i,j}{}'(\boldsymbol{v}_{ij}\cdot\hat{\boldsymbol{r}}_{ij})\,\hat{x}_{ij}\hat{y}_{ij}r_{ij}\Theta(d-r_{ij}).
\end{aligned} \tag{D6}
$$

From $\boldsymbol{v}_{ij} = \boldsymbol{p}_{ij}/m + \dot{\boldsymbol{\gamma}}\cdot\boldsymbol{r}_{ij}$ with $\dot{\gamma}_{\mu\nu} = \dot{\gamma}\delta_{\mu x}\delta_{\nu y}$, $\sigma_{xy}^{(\rm vis)}(\mathbf{\Gamma})$ can further be decomposed into

$$
\sigma_{xy}^{(\rm vis)}(\mathbf{\Gamma}) = \sigma_{xy}^{(\rm vis)\,(1)}(\mathbf{\Gamma}) + \sigma_{xy}^{(\rm vis)\,(2)}(\mathbf{\Gamma}), \tag{D7}
$$

where

$$
\sigma_{xy}^{(\rm vis)\,(1)}(\mathbf{\Gamma}) \equiv -\frac{\zeta}{2V}\sum_{i,j}{}'\left(\frac{\boldsymbol{p}_{ij}}{m}\cdot\hat{\boldsymbol{r}}_{ij}\right)\hat{x}_{ij}\hat{y}_{ij}r_{ij}\Theta(d-r_{ij}), \tag{D8}
$$

$$
\sigma_{xy}^{(\rm vis)\,(2)}(\mathbf{\Gamma}) \equiv -\frac{\dot{\gamma}\zeta}{2V}\sum_{i,j}{}'\hat{x}_{ij}^2\hat{y}_{ij}^2r_{ij}^2\Theta(d-r_{ij}). \tag{D9}
$$

Similarly, $\mathcal{R}(\mathbf{\Gamma})$ defined in Eq. (A8) can be decomposed into

$$
\mathcal{R}(\mathbf{\Gamma}) = \mathcal{R}^{(1)}(\mathbf{\Gamma}) + \mathcal{R}^{(2)}(\mathbf{\Gamma}) + \mathcal{R}^{(3)}(\mathbf{\Gamma}), \tag{D10}
$$

with

$$
\mathcal{R}^{(1)}(\mathbf{\Gamma}) \equiv \frac{\zeta}{4}\sum_{i,j}{}'\left(\frac{\boldsymbol{p}_{ij}}{m}\cdot\hat{\boldsymbol{r}}_{ij}\right)^2\Theta(d-r_{ij}), \tag{D11}
$$

$$
\mathcal{R}^{(2)}(\mathbf{\Gamma}) \equiv \frac{\dot{\gamma}\zeta}{2}\sum_{i,j}{}'\left(\frac{\boldsymbol{p}_{ij}}{m}\cdot\hat{\boldsymbol{r}}_{ij}\right)\hat{x}_{ij}\hat{y}_{ij}r_{ij}\Theta(d-r_{ij}), \tag{D12}
$$

$$
\mathcal{R}^{(3)}(\mathbf{\Gamma}) \equiv \frac{\dot{\gamma}^2\zeta}{4}\sum_{i,j}{}'\hat{x}_{ij}^2\hat{y}_{ij}^2r_{ij}^2\Theta(d-r_{ij}). \tag{D13}
$$

One easily recognizes from these expressions the following equalities,

$$\mathcal{R}^{(2)}(\boldsymbol{\Gamma}) = -\dot{\gamma} V \sigma_{xy}^{(\mathrm{vis})\,(1)}(\boldsymbol{\Gamma}), \tag{D14}$$

$$\mathcal{R}^{(3)}(\boldsymbol{\Gamma}) = -\frac{\dot{\gamma}}{2} V \sigma_{xy}^{(\mathrm{vis})\,(2)}(\boldsymbol{\Gamma}). \tag{D15}$$

Using these results, $\Omega_{\mathrm{eq}}(\boldsymbol{\Gamma})$ can be expressed as

$$\Omega_{\mathrm{eq}}(\boldsymbol{\Gamma}) = -\beta_{\mathrm{eq}}\left[\dot{\gamma} V \sigma_{xy}^{(\mathrm{el})}(\boldsymbol{\Gamma}) - \dot{\gamma} V \sigma_{xy}^{(\mathrm{vis})(1)}(\boldsymbol{\Gamma}) + 2\Delta\mathcal{R}_{\mathrm{eq}}^{(1)}(\boldsymbol{\Gamma})\right], \tag{D16}$$

where we have defined

$$\Delta\mathcal{R}_{\mathrm{eq}}^{(1)}(\boldsymbol{\Gamma}) \equiv \mathcal{R}^{(1)}(\boldsymbol{\Gamma}) + \frac{T_{\mathrm{eq}}}{2}\Lambda(\boldsymbol{\Gamma}). \tag{D17}$$

We attempt to obtain an approximate expression of Eq. (D1) in the form of the canonical distribution and its correction. The expression Eq. (D1) is a product of the canonical term and the exponential of the time integral of the work function. From Appendix C, the exponential factor can be rewritten as

$$\exp\left[\int_{-\infty}^{0} d\tau\, \Omega_{\mathrm{eq}}(\boldsymbol{\Gamma}(-\tau))\right] \approx e^{\tau_{\mathrm{rel}}\Omega_{\mathrm{SS}}(\boldsymbol{\Gamma})}, \tag{D18}$$

where τ_{rel} is the relaxation time, Eq. (C71). In Eq. (D18), we have defined

$$\Omega_{\mathrm{SS}}(\boldsymbol{\Gamma}) \equiv -\beta_{\mathrm{SS}}\left[\dot{\gamma} V \sigma_{xy}^{(\mathrm{el})}(\boldsymbol{\Gamma}) - \dot{\gamma} V \sigma_{xy}^{(\mathrm{vis})(1)}(\boldsymbol{\Gamma}) + 2\Delta\mathcal{R}_{\mathrm{SS}}^{(1)}(\boldsymbol{\Gamma})\right], \tag{D19}$$

which is equivalent to $\Omega_{\mathrm{eq}}(\boldsymbol{\Gamma})$ except that the inverse equilibrium temperature β_{eq} is replaced by its steady-state value β_{SS} via Eq. (A6). Accordingly, we replace β_{eq} in the canonical term of Eq. (D1) by β_{SS}. This gurantees the independence of the steady-state average

$$\langle \cdots \rangle_{\mathrm{SS}} \equiv \int d\boldsymbol{\Gamma}\, \rho_{\mathrm{SS}}(\boldsymbol{\Gamma}) \cdots \tag{D20}$$

on β_{eq}. As a result, we obtain

$$\rho_{\mathrm{SS}}(\boldsymbol{\Gamma}) = \frac{e^{-I_{\mathrm{SS}}(\boldsymbol{\Gamma})}}{\int d\boldsymbol{\Gamma} e^{-I_{\mathrm{SS}}(\boldsymbol{\Gamma})}}, \tag{D21}$$

where

$$I_{\mathrm{SS}}(\boldsymbol{\Gamma}) = \beta_{\mathrm{SS}}\mathcal{H}(\boldsymbol{\Gamma}) - \tau_{\mathrm{rel}}\Omega_{\mathrm{SS}}(\boldsymbol{\Gamma}). \tag{D22}$$

The steady-state temperature $T_{\mathrm{SS}} = \beta_{\mathrm{SS}}^{-1}$ is determined by the energy balance condition, which is given from Eq. (A9) as

$$\left\langle \dot{\mathcal{H}}(\boldsymbol{\Gamma}) \right\rangle_{\mathrm{SS}} = -\dot{\gamma} V \left\langle \sigma_{xy}(\boldsymbol{\Gamma}) \right\rangle_{\mathrm{SS}} - 2\left\langle \mathcal{R}(\boldsymbol{\Gamma}) \right\rangle_{\mathrm{SS}} = 0. \tag{D23}$$

Next, we consider the scaling with respect to ϵ and evaluate the order of magnitude of the terms in $\rho_{\mathrm{SS}}(\boldsymbol{\Gamma})$. The three terms in Eq. (D19) exhibit the following scaling properties,

$$\sigma_{xy}^{(\mathrm{el})*}(\boldsymbol{\Gamma}) = \epsilon^{1/2}\tilde{\sigma}_{xy}^{(\mathrm{el})}(\boldsymbol{\Gamma}), \tag{D24}$$

$$\sigma_{xy}^{(\mathrm{vis})(1)*}(\boldsymbol{\Gamma}) = \epsilon^{3/2}\tilde{\sigma}_{xy}^{(\mathrm{vis})(1)}(\boldsymbol{\Gamma}), \tag{D25}$$

$$\mathcal{R}^{(1)*}(\boldsymbol{\Gamma}) = \epsilon\,\tilde{\mathcal{R}}^{(1)}(\boldsymbol{\Gamma}), \tag{D26}$$

$$\Lambda^{*}(\boldsymbol{\Gamma}) = \epsilon\,\tilde{\Lambda}(\boldsymbol{\Gamma}), \tag{D27}$$

where the scaled quantites are given by

$$\tilde{\sigma}_{xy}^{(\mathrm{el})}(\boldsymbol{\Gamma}) = \frac{1}{V^*}\sum_{i=1}^{N}\left[\tilde{p}_{i,x}\tilde{p}_{i,y} + \tilde{y}_i \tilde{F}_{i,x}^{(\mathrm{el})}\right], \tag{D28}$$

$$\tilde{\sigma}_{xy}^{(\mathrm{vis})(1)}(\boldsymbol{\Gamma}) = -\frac{1}{2V^*}\sum_{i,j}{}' \left(\boldsymbol{p}_{ij}^* \cdot \hat{\boldsymbol{r}}_{ij}\right)\tilde{\hat{x}}_{ij}\tilde{\hat{y}}_{ij} r_{ij}^* \Theta(1 - r_{ij}^*), \tag{D29}$$

$$\tilde{\mathcal{R}}^{(1)}(\boldsymbol{\Gamma}) = \frac{1}{4}\sum_{i,j}{}' \left(\boldsymbol{p}_{ij}^* \cdot \hat{\boldsymbol{r}}_{ij}\right)^2 \Theta(1 - r_{ij}^*), \tag{D30}$$

$$\tilde{\Lambda}(\boldsymbol{\Gamma}) = -\sum_{i,j}{}' \Theta(1 - r_{ij}^*). \tag{D31}$$

From this, the order of magnitude of the three terms in $\Omega_{\mathrm{SS}}(\boldsymbol{\Gamma})$ can be evaluated as

$$\dot{\gamma}^* V^* \sigma_{xy}^{(\mathrm{el})*}(\boldsymbol{\Gamma}) = \epsilon\,\tilde{\dot{\gamma}} V^* \tilde{\sigma}_{xy}^{(\mathrm{el})}(\boldsymbol{\Gamma}), \tag{D32}$$

$$\dot{\gamma}^* V^* \sigma_{xy}^{(\mathrm{vis})(1)*}(\boldsymbol{\Gamma}) = \epsilon^2\,\tilde{\dot{\gamma}} V^* \tilde{\sigma}_{xy}^{(\mathrm{vis})(1)}(\boldsymbol{\Gamma}), \tag{D33}$$

$$\mathcal{R}^{(1)*}(\boldsymbol{\Gamma}) = \epsilon\,\tilde{\mathcal{R}}^{(1)}(\boldsymbol{\Gamma}). \tag{D34}$$

We retain $\sigma_{xy}^{(\mathrm{el})*}(\boldsymbol{\Gamma})$, $\mathcal{R}^{(1)*}(\boldsymbol{\Gamma})$ and neglect the higher-order term $\sigma_{xy}^{(\mathrm{vis})(1)*}(\boldsymbol{\Gamma})$, i.e.

$$\Omega_{\mathrm{SS}}^*(\boldsymbol{\Gamma}) \approx -\epsilon\,\beta_{\mathrm{SS}}^*\left[\tilde{\dot{\gamma}} V^* \tilde{\sigma}_{xy}^{(\mathrm{el})}(\boldsymbol{\Gamma}) + 2\Delta\tilde{\mathcal{R}}_{\mathrm{SS}}^{(1)}(\boldsymbol{\Gamma})\right]. \tag{D35}$$

As for the relaxation time, we introduce a scaled relaxation time $\tilde{\tau}_{\mathrm{rel}}$ with $\tau_{\mathrm{rel}}^* = \epsilon^{-1}\tilde{\tau}_{\mathrm{rel}}$, i.e.

$$\tilde{\tau}_{\mathrm{rel}} = \left[\frac{2\sqrt{\pi}}{3}\omega_E^*(T_{\mathrm{SS}}^*)\right]^{-1}. \tag{D36}$$

Note that $\tilde{\tau}_{\mathrm{rel}}$ is related to $\tilde{\dot{\gamma}}$ via T_{SS}^*, whose explicit expression will be given later in Eq. (E6). Then, $I_{\mathrm{SS}}(\boldsymbol{\Gamma})$ is approximated as

$$I_{\mathrm{SS}}^*(\boldsymbol{\Gamma}) \approx \beta_{\mathrm{SS}}^*\mathcal{H}^*(\boldsymbol{\Gamma}) - \tilde{\tau}_{\mathrm{rel}}\tilde{\Omega}_{\mathrm{SS}}(\boldsymbol{\Gamma}), \tag{D37}$$

where $\tilde{\Omega}_{\mathrm{SS}}(\boldsymbol{\Gamma}) = -\beta_{\mathrm{SS}}^*\left[\tilde{\dot{\gamma}} V^* \tilde{\sigma}_{xy}^{(\mathrm{el})}(\boldsymbol{\Gamma}) + 2\Delta\tilde{\mathcal{R}}_{\mathrm{SS}}^{(1)}(\boldsymbol{\Gamma})\right]$. We treat $\tilde{\tau}_{\mathrm{rel}}\tilde{\Omega}_{\mathrm{SS}}(\boldsymbol{\Gamma})$ as a correction, and expand $\rho_{\mathrm{SS}}(\boldsymbol{\Gamma})$ as

$$\rho_{\mathrm{SS}}(\boldsymbol{\Gamma}) \approx \frac{e^{-\beta_{\mathrm{SS}}^*\mathcal{H}^*(\boldsymbol{\Gamma})}\left[1 + \tilde{\tau}_{\mathrm{rel}}\tilde{\Omega}_{\mathrm{SS}}(\boldsymbol{\Gamma})\right]}{\mathcal{Z}} \tag{D38}$$

with $\mathcal{Z} \approx \int d\boldsymbol{\Gamma}\, e^{-\beta_{\mathrm{SS}}^*\mathcal{H}^*(\boldsymbol{\Gamma})}\left[1 + \tilde{\tau}_{\mathrm{rel}}\tilde{\Omega}_{\mathrm{SS}}(\boldsymbol{\Gamma})\right]$. An approximate expression for the ensemble average of an observable $A(\boldsymbol{\Gamma})$ with the weight $\rho_{\mathrm{SS}}(\boldsymbol{\Gamma})$ can be obtained from Eq. (D38) as

$$\langle A(\boldsymbol{\Gamma})\rangle_{\mathrm{SS}} \approx \langle A(\boldsymbol{\Gamma})\rangle_{\mathrm{eq}} + \tilde{\tau}_{\mathrm{rel}}\left\langle A(\boldsymbol{\Gamma})\tilde{\Omega}_{\mathrm{SS}}(\boldsymbol{\Gamma})\right\rangle_{\mathrm{eq}}, \tag{D39}$$

where $\langle \cdots \rangle_{\mathrm{eq}} = \int d\boldsymbol{\Gamma}\, e^{-\beta_{\mathrm{SS}}^*\mathcal{H}^*(\boldsymbol{\Gamma})} \cdots$ is the average with respect to the canonical distribution at the temperature T_{SS}.

Appendix E: Shear viscosity and temperature

The steady-state average of the shear stress and the energy dissipation rate is obtained by the formula Eq. (D39), from which we can derive an explicit expression for $T_{\rm SS}$. From the scaling arguments, the leading contribution to the shear stress comes from the elastic component $\sigma_{xy}^{(\rm el)}(\mathbf{\Gamma})$. Hence, $\langle \sigma_{xy}(\mathbf{\Gamma})\rangle_{\rm SS}$ is approximately given by

$$\langle \tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} \approx -\tilde{\tau}_{\rm rel}\tilde{\dot{\gamma}}\beta_{\rm SS}^* V^* \left\langle \tilde{\sigma}_{xy}^{(\rm el)}(\mathbf{\Gamma})\tilde{\sigma}_{xy}^{(\rm el)}(\mathbf{\Gamma})\right\rangle_{\rm eq}. \tag{E1}$$

This corresponds to the evaluation of the Green-Kubo formula by the relaxation time approximation in the correlation function. Similarly, the leading contribution to the energy dissipation rate comes from $\mathcal{R}^{(1)}(\mathbf{\Gamma})$ and hence we obtain

$$\left\langle \tilde{\mathcal{R}}(\mathbf{\Gamma})\right\rangle_{\rm SS} \approx \left\langle \tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})\right\rangle_{\rm eq} - 2\tilde{\tau}_{\rm rel}\beta_{\rm SS}^* \left\langle \tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})\Delta\tilde{\mathcal{R}}_{\rm SS}^{(1)}(\mathbf{\Gamma})\right\rangle_{\rm eq}. \tag{E2}$$

In these expressions, the anisotropic terms $\langle \tilde{\sigma}_{xy}^{(\rm el)}(\mathbf{\Gamma})\rangle_{\rm eq}$ and $\langle \tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})\tilde{\sigma}_{xy}^{(\rm el)}(\mathbf{\Gamma})\rangle_{\rm eq}$ are identically zero. We obtain the following results in the hard-core limit from straightforward calculations,

$$\left\langle \tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})\right\rangle_{\rm eq} = \frac{\sqrt{\pi}}{2} N T_{\rm SS}^* \omega_E^*(T_{\rm SS}^*), \tag{E3}$$

$$\left\langle \tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})\Delta\tilde{\mathcal{R}}_{\rm SS}^{(1)}(\mathbf{\Gamma})\right\rangle_{\rm eq} = \frac{1}{8} N T_{\rm SS}^{*2} \frac{\omega_E(T_{\rm SS}^*)^{*2}}{n^* g_0(\varphi)} \times \left[\mathscr{R}_2 + \mathscr{R}_3\, n^* g_0(\varphi)\right], \tag{E4}$$

$$\left\langle \tilde{\sigma}_{xy}^{(\rm el)}(\mathbf{\Gamma})\tilde{\sigma}_{xy}^{(\rm el)}(\mathbf{\Gamma})\right\rangle_{\rm eq} = \frac{1}{V^*} n^* T_{\rm SS}^{*2} \times \left[1 + \mathscr{S}_2\, n^* g_0(\varphi) + \mathscr{S}_3\, n^{*2} g_0(\varphi)^2 + \mathscr{S}_4\, n^{*3} g_0(\varphi)^3\right], \tag{E5}$$

where $\mathscr{R}_2 = 1$, $\mathscr{R}_3 = 3\pi/4$, $\mathscr{S}_2 = 2\pi/15$, $\mathscr{S}_3 = -\pi^2/20$, and $\mathscr{S}_4 = 3\pi^3/160$. Explicit calculations of the equilibrium correlations, Eqs. (E3)–(E5), are shown in Sec. F in detail. To be specific, Eq. (E3) is derived from Eq. (F1), Eq. (E4) from Eqs. (F10) and (F11), and Eq. (E5) from (F14), (F22), and (F31). Here, the terms with coefficients $\mathscr{R}_i$ and $\mathscr{S}_i$ are contributions of the i-body correlations, and the first term of Eq. (E5) is the contribution of the kinetic stress. From the energy balance Eq. (D23) of order ϵ, i.e. $-\tilde{\dot{\gamma}}V^* \langle \tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} - 2\left\langle \tilde{\mathcal{R}}(\mathbf{\Gamma})\right\rangle_{\rm SS} = 0$, we obtain the steady-state temperature $T_{\rm SS}$ as

$$T_{\rm SS}^* = \frac{3\tilde{\dot{\gamma}}^2}{32\pi}\frac{S}{R}, \tag{E6}$$

where S and R are given by

$$S = 1 + \mathscr{S}_2\, n^* g_0(\varphi) + \mathscr{S}_3\, n^{*2} g_0(\varphi)^2 + \mathscr{S}_4\, n^{*3} g_0(\varphi)^3, \tag{E7}$$

$$R = n^* g_0(\varphi)\left[\mathscr{R}_2' + \mathscr{R}_3'\, n^* g_0(\varphi)\right], \tag{E8}$$

with $\mathscr{R}_2' = -3/4$, $\mathscr{R}_3' = 7\pi/16$. We further obtain the expression for the shear stress,

$$\langle \tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} = -\frac{3}{8\pi}\tilde{\dot{\gamma}}\, T_{\rm SS}^{*1/2}\frac{S}{g_0(\varphi)} = -\frac{3\sqrt{6}}{64\pi^{3/2}}\tilde{\dot{\gamma}}^2 \frac{S^{3/2}}{R^{1/2} g_0(\varphi)}. \tag{E9}$$

In the vicinity of the jamming point φ_J, S and R can be approximated as $S \approx \mathscr{S}_4\, n^{*3} g_0(\varphi)^3$ and $R \approx \mathscr{R}_3'\, n^{*2} g_0(\varphi)^2$, respectively. This leads to the following approximate expressions,

$$T_{\rm SS}^* \approx \frac{3\tilde{\dot{\gamma}}^2}{32\pi}\frac{\mathscr{S}_4}{\mathscr{R}_3'} n^* g_0(\varphi) = \frac{9\pi}{2240}\tilde{\dot{\gamma}}^2 n^* g_0(\varphi), \tag{E10}$$

$$\langle \tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} \approx -\frac{9\pi^2}{1280}\tilde{\dot{\gamma}}\, {T_{\rm SS}^*}^{1/2} n^{*3} g_0(\varphi)^2 = -\frac{27\pi^{5/2}}{10240\sqrt{35}}\tilde{\dot{\gamma}}^2 n^{*7/2} g_0(\varphi)^{5/2}. \tag{E11}$$

In the course of the derivation, we have utilized Eq. (D36). Note that this expression agrees with the scaling from the dimensional analysis and obeys the Bagnold scaling, $\langle \tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} \propto \tilde{\dot{\gamma}}^2$. From these expressions, we obtain the scalings

$$T_{\rm SS}^* \sim g_0(\varphi) \sim (\varphi_J - \varphi)^{-1}, \tag{E12}$$

$$\langle \tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS} \sim {T_{\rm SS}^*}^{1/2} g_0(\varphi)^2 \sim (\varphi_J - \varphi)^{-5/2}. \tag{E13}$$

From Eq. (E9), we obtain the expression for the shear viscosity $\eta^* = -\langle \tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS}/\tilde{\dot{\gamma}}$,

$$\eta^* \approx \frac{27\pi^{5/2}}{10240\sqrt{35}}\dot{\gamma}^* n^{*7/2} g_0(\varphi)^{5/2} \sim (\varphi_J - \varphi)^{-5/2}, \tag{E14}$$

or for $\tilde{\eta}' = -\langle \tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS}/(\tilde{\dot{\gamma}}\sqrt{T_{\rm SS}^*}) \propto -\langle \tilde{\sigma}_{xy}(\mathbf{\Gamma})\rangle_{\rm SS}/\tilde{\dot{\gamma}}^2$,

$$\tilde{\eta}' \approx \frac{9\pi^2}{1280} n^{*3} g_0(\varphi)^2 \sim (\varphi_J - \varphi)^{-2}. \tag{E15}$$

Appendix F: Equilibrium correlations

We derive the equilibrium correlations, Eqs. (E3)–(E5). In the course of the derivation, the hard-core limit is taken with the aid of Eqs. (G6) and (G12).

First, Eq. (E3) is calculated as

$$\begin{aligned}\left\langle \tilde{\mathcal{R}}^{(1)}(\mathbf{\Gamma})\right\rangle_{\rm eq} &= \frac{1}{4}\left\langle {\sum_{i,j}}' (\boldsymbol{p}_{ij}^* \cdot \hat{\boldsymbol{r}}_{ij})^2 \Theta(1 - r_{ij}^*)\right\rangle_{\rm eq} \\ &= \frac{1}{2}T_{\rm SS}^*\left\langle {\sum_{i,j}}' \Theta(1 - r_{ij}^*)\right\rangle_{\rm eq} \\ &\approx \frac{1}{2}N n^* T_{\rm SS}^* \int d^3\boldsymbol{r}^*\, g(r^*)\Theta(1 - r^*) \\ &\approx \frac{\sqrt{\pi}}{2} N T_{\rm SS}^* \omega_E^*(T_{\rm SS}^*),\end{aligned} \tag{F1}$$

where we have utilized $\langle {\sum_{i,j}}' \delta(\boldsymbol{r}^* - \boldsymbol{r}_{ij}^*)\rangle_{\rm eq} \approx N n^* g(r^*)$ with $g(r)$ the radial distribution function at equilibrium in the third equality and Eq. (G12) in the last equality.

Next, we deal with Eq. (E4),

$$\left\langle \tilde{\mathcal{R}}^{(1)}\Delta\tilde{\mathcal{R}}_{\rm SS}^{(1)}\right\rangle_{\rm eq} = \left\langle \tilde{\mathcal{R}}^{(1)}\tilde{\mathcal{R}}^{(1)}\right\rangle_{\rm eq} + \frac{T_{\rm SS}^*}{2}\left\langle \tilde{\mathcal{R}}^{(1)}\tilde{\Lambda}\right\rangle_{\rm eq}. \tag{F2}$$

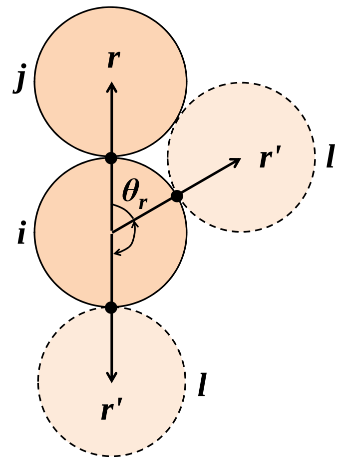

FIG. 3. (Color online) Schematic figure for the three-body correlation. The angle θ_r between $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{r}}'$ is in the range $[\pi/3,\ \pi]$.

The first term is decomposed as

$$\left\langle \tilde{\mathcal{R}}^{(1)}\tilde{\mathcal{R}}^{(1)}\right\rangle_{\rm eq} = \frac{T_{\rm SS}^{*2}}{4}\left\langle \sum_{i,j}{}'\sum_{l,k}{}'\tilde{\Lambda}(\boldsymbol{r}_{ij}^*)\tilde{\Lambda}(\boldsymbol{r}_{lk}^*)\right\rangle_{\rm eq}$$
$$+8T_{\rm SS}^{*2}\left\langle \sum_{i,j,l}{}''\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}_{ij}^*)\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}_{il}^*)\right\rangle_{\rm eq}$$
$$+8T_{\rm SS}^{*2}\left\langle \sum_{i,j}{}'\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}_{ij}^*)\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}_{ij}^*)\right\rangle_{\rm eq}, \quad \text{(F3)}$$

where we have defined $\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}^*) = \Theta(1-r^*)\hat{r}^\mu\hat{r}^\nu/4$. Here, $\sum_{i,j,l}{}''$ is performed under the condition that any two pair of particles (i,j,l) is different. On the other hand, the second term on the right hand side of Eq. (F2) is given by

$$\frac{T_{\rm SS}^*}{2}\left\langle \tilde{\mathcal{R}}^{(1)}\tilde{\Lambda}\right\rangle_{\rm eq} = -\frac{T_{\rm SS}^{*2}}{4}\left\langle \sum_{i,j}{}'\sum_{l,k}{}'\tilde{\Lambda}(\boldsymbol{r}_{ij}^*)\tilde{\Lambda}(\boldsymbol{r}_{lk}^*)\right\rangle_{\rm eq}. \quad \text{(F4)}$$

Hence, the four-body correlation cancels out and the two- and three-body correlations remain.

The three-body correlation is calculated as

$$\left\langle \sum_{i,j,l}{}''\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}_{ij}^*)\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}_{il}^*)\right\rangle_{\rm eq}$$
$$= \frac{1}{16}\int d^3\boldsymbol{r}^*\int d^3\boldsymbol{r}'^*\left\langle \sum_{i,j,l}{}''\delta(\boldsymbol{r}^*-\boldsymbol{r}_{ij}^*)\delta(\boldsymbol{r}'^*-\boldsymbol{r}_{il}^*)\right\rangle_{\rm eq}$$
$$\times(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}')^2\,\Theta(1-r^*)\Theta(1-r'^*)$$
$$\approx \frac{Nn^{*2}}{16}\int d^3\boldsymbol{r}^*\int d^3\boldsymbol{r}'^* g^{(3)}(\boldsymbol{r}^*,\boldsymbol{r}'^*)\,(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}')^2$$
$$\times\Theta(1-r^*)\Theta(1-r'^*), \quad \text{(F5)}$$

where $g^{(3)}(\boldsymbol{r},\boldsymbol{r}')$ is the triple correlation function at equilibrium. The triple correlation is conventionally approximated by the Kirkwood's superposition approximation as $g^{(3)}(\boldsymbol{r},\boldsymbol{r}')\approx g(r)g(r')g(|\boldsymbol{r}-\boldsymbol{r}'|)$, where $g(r)$ and $g(r')$ are the radial correlations and $g(|\boldsymbol{r}-\boldsymbol{r}'|)$ is the angular correlation [78] (cf. Fig. 3). In the present case, we are interested in the case where the spheres $(i,\ j)$ and $(i,\ l)$ are in contact. To ensure the connectivity, we insert a step function as

$$g^{(3)}(\boldsymbol{r},\boldsymbol{r}')\approx g(r)\Theta(d-r)g(r')\Theta(d-r')g(|\boldsymbol{r}-\boldsymbol{r}'|), \quad \text{(F6)}$$

which leads to

$$\left\langle \sum_{i,j,l}{}''\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}_{ij}^*)\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}_{il}^*)\right\rangle_{\rm eq}$$
$$\approx\frac{Nn^{*2}}{16}\int_0^\infty dr^*\int_0^\infty dr'^* g(r^*)g(r'^*)r^{*2}r'^{*2}$$
$$\times\Theta(1-r^*)^2\Theta(1-r'^*)^2$$
$$\times\int d\mathcal{S}\int d\mathcal{S}'\,(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}')^2\,g(|\boldsymbol{r}^*-\boldsymbol{r}'^*|). \quad \text{(F7)}$$

Here, $\int d\mathcal{S}$ and $\int d\mathcal{S}'$ represent the angular integrations with respect to the solid angles of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{r}}'$, respectively. It should be noted that there is a subtlety in the integral of the angular correlation, $g(|\boldsymbol{r}^*-\boldsymbol{r}'^*|)$, since spheres j and l are in contact when $\theta_r=\pi/3$, where $\cos\theta_r\equiv\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}'$. The radial distribution function $g(r)=1+h(r)$, where $h(r)$ is the total correlation function, consists of a δ-function (contact) contribution at $r\approx d$ and a regular contribution, which is approximately 1. Hence, it is reasonable to approximate $h(r)$ by the δ-function contribution, which is given by

$$h(r)\approx\frac{1}{4\varphi\delta}\left[\frac{6A}{\left(\frac{r/d-1}{\delta}+C\right)^4}+\frac{B}{\left(\frac{r/d-1}{\delta}+C\right)^2}\right], \quad \text{(F8)}$$

with numerically fitted coefficients, $A\approx3.43$, $B\approx1.45$, and $C\approx2.25$ [79]. Here, $\delta>0$ is defined by $\varphi=\varphi_J(1-\delta)^3$, which is approximated as $\delta\approx(\varphi_J-\varphi)/(3\varphi_J)$ for $\varphi\approx\varphi_J$. The angular integral is evaluated as (cf. Fig. 3)

$$\int d\mathcal{S}\int d\mathcal{S}'\,(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}')^2\,[1+h(|\boldsymbol{r}^*-\boldsymbol{r}^{*\prime}|)]$$
$$=8\pi^2\int_{-1}^{1/2}d(\cos\theta_r)\cos^2\theta_r\left[1+h\left(\sqrt{2(1-\cos\theta_r)}\right)\right]$$
$$=3\pi^2+\int_0^1 dz\,(z+1)\left[1-\frac{1}{2}(z+1)^2\right]^2\hat{h}(z), \quad \text{(F9)}$$

where $\hat{h}(z) = \left[6A/(z/\delta+C)^4+(z/\delta+C)^2\right]/(4\varphi\delta)$. Note that θ_r is restricted to $\theta_r\in[\pi/3,\pi]$ due to the constraint that any two pair of particles (i,j,l) is different. In approaching the jamming point, i.e. $\delta\to0$, $\hat{h}(z)$ behaves as $\hat{h}(z)\sim\delta^{-1}\left[(\delta/z)^4+(\delta/z)^2\right]\to0$ and hence does not contribute to the angular integral. The radial

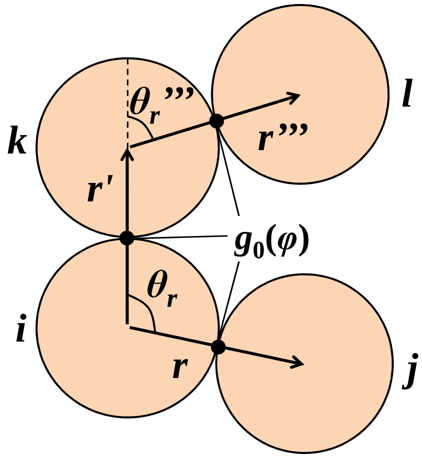

FIG. 4. (Color online) Schematic figure for the four-body correlation.

integrations can be carried out with the aid of Eq. (G12), from which we obtain

$$\left\langle \sum_{i,j,l}{}'' \tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}^*_{ij})\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}^*_{il}) \right\rangle_{\rm eq} \approx \frac{3\pi}{256} N\omega_E^{*2}(T^*_{\rm SS}). \tag{F10}$$

The two-body correlation is calculated, similarly to Eq. (F1), as

$$\begin{aligned} &\left\langle \sum_{i,j}{}' \tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}^*_{ij})\tilde{\mathcal{R}}^{(1)\mu\nu}(\boldsymbol{r}^*_{ij}) \right\rangle_{\rm eq} \\ &= \frac{1}{16}\int d^3\boldsymbol{r}^* \left\langle \sum_{i,j}{}' \delta(\boldsymbol{r}^*-\boldsymbol{r}^*_{ij}) \right\rangle_{\rm eq} \Theta(1-r^*)^2 \\ &\approx \frac{Nn^*}{16} g_0(\varphi)\int d^3\boldsymbol{r}^* \Xi(r^*)^2 = \frac{1}{64}N\frac{\omega_E^{*2}(T^*_{\rm SS})}{n^* g_0(\varphi)}, \end{aligned} \tag{F11}$$

where Eq. (G12) is utilized in the last equality. This concludes the derivation of Eq. (E4).

Finally, we deal with Eq. (E5). From the definition of the elastic stress tensor, Eq. (D28), we obtain

$$\begin{aligned} \left\langle \tilde{\sigma}^{(\rm el)}_{xy}\tilde{\sigma}^{(\rm el)}_{xy} \right\rangle_{\rm eq} &= \frac{1}{V^{*2}} \Bigg\{ NT^{*2}_{\rm SS} \\ &+\frac{1}{4}\left\langle \sum_{i,j}{}' \sum_{l,k}{}' y^*_{ij}y^*_{lk}F^{({\rm el})x*}_{ij}F^{({\rm el})x*}_{lk} \right\rangle_{\rm eq} \Bigg\}, \end{aligned} \tag{F12}$$

where the first term is the kinetic stress and the second term is the contact stress. The contact stress consists of three components, i.e. the two-, three-, and four-body correlations.

First, the two-body correlation is calculated as

$$\begin{aligned} &\frac{1}{2}\left\langle \sum_{i,j}{}' y^{*2}_{ij}F^{({\rm el})x*2}_{ij} \right\rangle_{\rm eq} \approx \frac{1}{2}Nn^*\int d^3\boldsymbol{r}^* g(r^*) y^{*2}F^{({\rm el})*2}_x \\ &= \frac{1}{2}Nn^*\int_0^\infty dr^* g(r^*)F^{({\rm el})*2}r^{*4}\int d\mathcal{S}\, \hat{x}^2\hat{y}^2 \\ &= \frac{2\pi}{15}Nn^*\int_0^\infty dr^* g(r^*)F^{({\rm el})*2}r^{*4}, \end{aligned} \tag{F13}$$

where $F^{({\rm el})*}(r^*) = -\Theta(1-r^*)\,u^{*\prime}(r^*)$ is the magnitude of the elastic force with $u^*(r^*)$ the pair potential, and $\int d\mathcal{S}\cdots$ is the integration with respect to the solid angle of $\hat{\boldsymbol{r}}$. From Eq. (G6), we we obtain the following expression in the hard-core limit,

$$\frac{1}{2}\left\langle \sum_{i,j}{}' y^{*2}_{ij}F^{({\rm el})x*2}_{ij} \right\rangle_{\rm eq} \approx \frac{2\pi}{15}NT^{*2}_{\rm SS}n^*g_0(\varphi). \tag{F14}$$

Next, the three-body correlation is calculated as

$$\begin{aligned} &\frac{1}{2}\left\langle \sum_{i,j,k}{}'' y^*_{ij}y^*_{ik}F^{({\rm el})x*}_{ij}F^{({\rm el})x*}_{ik} \right\rangle_{\rm eq} \\ &\approx \frac{1}{2}Nn^{*2}\int d^3\boldsymbol{r}^*\int d^3\boldsymbol{r}'^* g^{(3)}(\boldsymbol{r}^*,\boldsymbol{r}'^*) r^*r'^* \\ &\quad\times \hat{x}\hat{y}\hat{x}'\hat{y}' F^{({\rm el})*}(r^*)F^{({\rm el})*}(r'^*) \\ &\approx \frac{1}{2}Nn^{*2}\int_0^\infty dr^*\int_0^\infty dr'^* g(r^*)g(r'^*)r^{*3}r'^{*3} \\ &\quad\times F^{({\rm el})*}(r^*)F^{({\rm el})*}(r'^*)\Theta(1-r^*)\Theta(1-r'^*) \\ &\quad\times\int d\mathcal{S}\int d\mathcal{S}' \hat{x}\hat{y}\hat{x}'\hat{y}'\, g(|\boldsymbol{r}^*-\boldsymbol{r}^{*\prime}|), \end{aligned} \tag{F15}$$

where Eq. (F6) is applied in the second equality. Similarly to Eq. (F7), the δ-function contribution of $g(|\boldsymbol{r}^*-\hat{\boldsymbol{r}}^{*\prime}|)$ vanishes in the angular integrals. Hence, we obtain

$$\begin{aligned} &\int d\mathcal{S}\int d\mathcal{S}'\, \hat{x}\hat{y}\hat{x}'\hat{y}'\, g(|\boldsymbol{r}^*-\boldsymbol{r}^{*\prime}|) \\ &= \int_{-1}^{\frac{1}{2}} d(\cos\theta_r)\int_{-1}^1 d(\cos\theta')\int_0^{2\pi} d\phi_r\int_0^{2\pi} d\phi' \hat{x}\hat{y}\hat{x}'\hat{y}' \\ &= -\frac{\pi^2}{10}. \end{aligned} \tag{F16}$$

Here, $\cos\theta_r \equiv \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}'$ and θ', ϕ' are defined as

$$\hat{x}' = \sin\theta'\cos\phi', \tag{F17}$$
$$\hat{y}' = \sin\theta'\sin\phi', \tag{F18}$$
$$\hat{z}' = \cos\theta'. \tag{F19}$$

This gives $\hat{x}$ and $\hat{y}$ in terms of θ', ϕ', θ_r, and ϕ_r as

$$\begin{aligned} \hat{x} &= \left(1-\frac{\hat{x}'^2}{1+\hat{z}'}\right)\sin\theta_r\cos\phi_r - \frac{\hat{x}'\hat{y}'}{1+\hat{z}'}\sin\theta_r\sin\phi_r \\ &\quad+\hat{x}'\cos\theta_r, \end{aligned} \tag{F20}$$

$$\begin{aligned} \hat{y} &= -\frac{\hat{x}'\hat{y}'}{1+\hat{z}'}\sin\theta_r\cos\phi_r + \left(1-\frac{\hat{y}'^2}{1+\hat{z}'}\right)\sin\theta_r\sin\phi_r \\ &\quad+\hat{y}'\cos\theta_r, \end{aligned} \tag{F21}$$

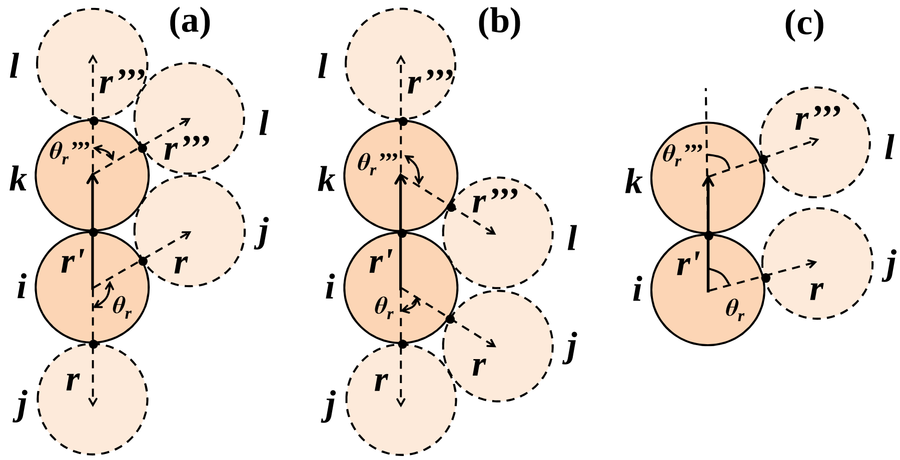

FIG. 5. (Color online) Schematic figure for the four-body correlation.

where ϕ_r is the azimuthal angle of $\hat{\boldsymbol{r}}$ in the spherical coordinate where $\hat{\boldsymbol{r}}'$ points in the z-direction. Note that θ_r is restricted to $\theta_r \in [\pi/3, \pi]$, similarly to Eq. (F9). The radial integrations can be carried out with the aid of Eq. (G6), from which we obtain

$$\frac{1}{2}\left\langle {\sum_{i,j,k}}'' y^*_{ij} y^*_{ik} F^{(\mathrm{el})x*}_{ij} F^{(\mathrm{el})x*}_{ik} \right\rangle_{\mathrm{eq}} \approx -\frac{\pi^2}{20} N T^{*2}_{\mathrm{SS}} n^{*2} g_0(\varphi)^2. \tag{F22}$$

Finally, the four-body correlation is calculated as

$$\begin{aligned} &\frac{1}{4}\left\langle {\sum_{i,j,l,k}}''' y^*_{ij} y^*_{lk} F^{(\mathrm{el})x*}_{ij} F^{(\mathrm{el})x*}_{lk} \right\rangle_{\mathrm{eq}} \\ &= \frac{1}{4}\left\langle {\sum_{i,j,l,k}}''' y^*_{ij} (y^*_{ik} - y^*_{il}) F^{(\mathrm{el})x*}_{ij} F^{(\mathrm{el})x*}_{lk} \right\rangle_{\mathrm{eq}} \\ &\approx \frac{1}{2} N n^{*3} \int d^3\boldsymbol{r}^* \int d^3\boldsymbol{r}'^* \int d^3\boldsymbol{r}''^* g^{(4)}(\boldsymbol{r}^*, \boldsymbol{r}'^*, \boldsymbol{r}''^*) \\ &\times y^* (y'^* - y''^*) \hat{x} (\hat{x}' - \hat{x}'') F^{(\mathrm{el})*}(r^*) F^{(\mathrm{el})*}(|\boldsymbol{r}'^* - \boldsymbol{r}''^*|), \end{aligned} \tag{F23}$$

where $g^{(4)}(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{r}'')$ is the quadruple correlation at equilibrium. Here, $\sum_{i,j,l,k}'''$ is performed under the condition that any two pair of particles (i, j, l, k) is different. We change the integration variable from $\boldsymbol{r}''^*$ to $\boldsymbol{r}'''^* \equiv \boldsymbol{r}''^* - \boldsymbol{r}'^*$, which leads to

$$\begin{aligned} &\frac{1}{4}\left\langle {\sum_{i,j,l,k}}''' y^*_{ij} y^*_{lk} F^{(\mathrm{el})x*}_{ij} F^{(\mathrm{el})x*}_{lk} \right\rangle_{\mathrm{eq}} \\ &= \frac{1}{2} N n^{*3} \int d^3\boldsymbol{r}^* \int d^3\boldsymbol{r}'^* \int d^3\boldsymbol{r}'''^* g^{(4)}(\boldsymbol{r}^*, \boldsymbol{r}'^*, \boldsymbol{r}'''^*) \\ &\times y^* y'''^* \hat{x} \hat{x}''' F^{(\mathrm{el})*}(r^*) F^{(\mathrm{el})*}(r'''^*). \end{aligned} \tag{F24}$$

Similarly to Eq. (F6), we adopt the following approximation,

$$\begin{aligned} g^{(4)}(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{r}''') &\approx g(r)\Theta(d-r) g(r')\Theta(d-r') g(r''')\Theta(d-r''') \\ &\times g(|\boldsymbol{r}-\boldsymbol{r}'|) g(|\boldsymbol{r}'+\boldsymbol{r}'''|) g(|\boldsymbol{r}'+\boldsymbol{r}'''-\boldsymbol{r}|), \end{aligned} \tag{F25}$$

where the step function is inserted to ensure the connectivity of the spheres (i, j), (k, l), and (i, k). Then, Eq. (F24) is approximated as

$$\begin{aligned} &\frac{1}{4}\left\langle {\sum_{i,j,l,k}}''' y^*_{ij} y^*_{lk} F^{(\mathrm{el})x*}_{ij} F^{(\mathrm{el})x*}_{lk} \right\rangle_{\mathrm{eq}} \\ &\approx \frac{1}{2} N n^{*3} \int_0^\infty dr^* \int_0^\infty dr'^* \int_0^\infty dr'''^* g(r^*) g(r'^*) g(r'''^*) \\ &\times r^{*3} r'^{*2} r'''^{*3} F^{(\mathrm{el})*}(r^*) F^{(\mathrm{el})*}(r'''^*) \\ &\times \Theta(1-r^*)\Theta(1-r'^*)\Theta(1-r'''^*) \\ &\times \int d\mathcal{S} \int d\mathcal{S}' \int d\mathcal{S}''' \hat{x}\hat{y}\hat{x}'''\hat{y}''' \\ &\times g(|\boldsymbol{r}^* - \boldsymbol{r}'^*|) g(|\boldsymbol{r}'^* + \boldsymbol{r}'''^*|) g(|\boldsymbol{r}'^* + \boldsymbol{r}'''^* - \boldsymbol{r}^*|). \end{aligned} \tag{F26}$$

Here, $\int d\mathcal{S}$, $\int d\mathcal{S}'$, $\int d\mathcal{S}'''$ represent the angular integrations with respect to the solid angles of $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{r}}'$, $\hat{\boldsymbol{r}}'''$, respectively.

We first consider the radial integration in Eq. (F26). From Eq. (G6), integrations with r^* and r'''^* give $T^{*2}_{\mathrm{SS}}\, g_0(\varphi)^2$. On the other hand, the integration with r'^* gives $g_0(\varphi)$. Hence, the radial integration is given by $T^{*2}_{\mathrm{SS}} g_0(\varphi)^3$.

Next we consider the angular integration in Eq. (F26). Similarly to Eq. (F7), the δ-function contribution of the raidal correlations $g(|\boldsymbol{r}^* - \boldsymbol{r}'^*|)$, $g(|\boldsymbol{r}'^* + \boldsymbol{r}'''^*|)$, and $g(|\boldsymbol{r}'^* + \boldsymbol{r}'''^* - \boldsymbol{r}^*|)$ vanishes. Hence, it is given by

$$\begin{aligned} &\int d\mathcal{S} \int d\mathcal{S}' \int d\mathcal{S}''' \hat{x}\hat{y}\hat{x}'''\hat{y}''' \\ &= \int_{-1}^{1} d(\cos\theta') \int d(\cos\theta_r) \int d(\cos\theta'''_r) \\ &\times \int_0^{2\pi} d\phi' \int d\phi_r \int d\phi'''_r\, \hat{x}\hat{y}\hat{x}'''\hat{y}''', \end{aligned} \tag{F27}$$

where $\cos\theta_r \equiv \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'$, $\cos\theta'''_r \equiv \hat{\boldsymbol{r}}''' \cdot \hat{\boldsymbol{r}}'$, and θ', ϕ' are defined by Eqs. (F17)–(F19). Then, $\hat{x}$ and $\hat{y}$ are given by Eqs. (F20) and (F21), and $\hat{x}'''$ and $\hat{y}'''$ are given in

terms of θ_r''', ϕ_r''', θ', and ϕ' by the replacement $\theta_r \to \theta_r'''$, $\phi_r \to \phi_r'''$ in Eqs. (F20) and (F21). For the case (a), where $\theta_r \in [\pi/3, \pi]$ and $\theta_r''' \in [0, \pi/3]$ (cf. Fig. 5), sphere j and l can move freely without interference in the ϕ_r and ϕ_r''' direction, i.e. $\phi_r \in [0, 2\pi]$ and $\phi_r''' \in [0, 2\pi]$. Then, the angular integral is evaluated as

$$\int_{-1}^{\frac{1}{2}} d(\cos\theta_r) \int_{\frac{1}{2}}^{1} d(\cos\theta_r''')\, f(\theta_r, \theta_r''') = -\frac{3\pi^3}{80}, \quad \text{(F28)}$$

where $f(\theta_r, \theta_r''') \equiv 4\pi^3(3\cos^2\theta_r - 1)(3\cos^2\theta_r''' - 1)/15$ is the result of the integration with respect to θ', ϕ', ϕ_r, and ϕ_r'''. Similarly, for the case (b), where $\theta_r \in [2\pi/3, \pi]$ and $\theta_r''' \in [0, 2\pi/3]$ (cf. Fig. 5), the angular integral is evaluated as

$$\int_{-1}^{-\frac{1}{2}} d(\cos\theta_r) \int_{-\frac{1}{2}}^{1} d(\cos\theta_r''') f(\theta_r, \theta_r''') = -\frac{3\pi^3}{80}. \quad \text{(F29)}$$

As for the case (c), where θ_r and θ_r''' vary in the range $\theta_r \in [\pi/3, 2\pi/3]$, $\theta_r''' \in [\pi/3, \theta_r]$ (cf. Fig. 5), we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d(\cos\theta_r) \int_{\cos\theta_r}^{\frac{1}{2}} d(\cos\theta_r''') f(\theta_r, \theta_r''') = \frac{9\pi^3}{80}. \quad \text{(F30)}$$

Hence, the angular integration is given by $3\pi^3/80$. Together with the radial integration, we obtain

$$\frac{1}{4}\left\langle \sum_{i,j,l,k}{}''' y_{ij}^* y_{lk}^* F_{ij}^{(\mathrm{el})x*} F_{lk}^{(\mathrm{el})x*} \right\rangle_{\mathrm{eq}} \approx \frac{3\pi^3}{160} N T_{\mathrm{SS}}^{*2} n^{*3} g_0(\varphi)^3. \quad \text{(F31)}$$

This concludes the derivation of Eq. (E5).

Appendix G: Hard-core limit

In systems of soft spheres with contact interactions, the Heaviside's step function, $\Theta(d-r)$, appears in the interparticle forces as in Eqs. (A2) and (A3). This step function reduces to a delta function, $\delta(d-r)$, in the hard-core limit. Here, we derive formulas valid in this limit.

First, we derive a formula for the step function in the elastic force. It is convenient to introduce a function which is referred to as the cavity distribution function [78],

$$y(r) = g(r)\, e(r)^{-1}, \quad \text{(G1)}$$

where

$$e(r) = e^{-\beta_{\mathrm{SS}}\, u(r)} \quad \text{(G2)}$$

with $u(r) = (\kappa/2)\,\Theta(d-r)(d-r)^2$ the pair potential of the elastic force. It should be noted that $e(r)$ reduces to the step function in the hard-core limit, $\kappa d^2/T_{\mathrm{SS}} \to \infty$, and hence its derivative $e'(r)$ reduces to the delta function, $e'(r) \to \delta(d-r)$, or $e'(r^*) \to \delta(1-r^*)$ in the non-dimensionalized form. Let us consider the hard-core limit of

$$\int_0^\infty dr^* g(r^*) F^{(\mathrm{el})*} r^{*2}, \quad \text{(G3)}$$

where $F^{(\mathrm{el})*}(r^*) = -\Theta(1-r^*)\, u^{*\prime}(r^*)$ is the magnitude of the elastic force with $u^*(r^*)$ the pair potential, for illustration. From $u'(r)e(r) = -T_{\mathrm{SS}}\, e'(r)$, we obtain the following expression

$$\int_0^\infty dr^* g(r^*) F^{(\mathrm{el})*} r^{*2} = -\int_0^\infty dr^* g(r^*)\Theta(1-r^*) u^{*\prime}(r^*) r^{*2}$$
$$= -T_{\mathrm{SS}}^* \int_0^\infty dr^* g(r^*) e'(r^*) r^{*2} \to -T_{\mathrm{SS}}^*\, g_0(\varphi). \quad \text{(G4)}$$

This corresponds to the replacement

$$F^{(\mathrm{el})}(r) g(r) \to -T_{\mathrm{SS}}\, \delta(d-r) g_0(\varphi), \quad \text{(G5)}$$

or, in non-dimensionalized form,

$$F^{(\mathrm{el})*}(r^*) g(r^*) \to -T_{\mathrm{SS}}^*\, \delta(1-r^*) g_0(\varphi). \quad \text{(G6)}$$

Next, we derive a formula for the step function in the viscous force Eq. (A3), which appears in e.g. Eqs. (B9), (D8), and (D11). In accordance with the elastic force, it is convenient to introduce a variant of the cavity distribution function,

$$y_{1/2}(r) = g(r)\, e_{1/2}(r)^{-1}, \quad \text{(G7)}$$

where

$$e_{1/2}(r) = e^{-\sqrt{2\beta_{\mathrm{SS}}\, u(r)}}. \quad \text{(G8)}$$

Similarly to Eq. (G2), $e_{1/2}(r)$ reduces to the step function $\Theta(d-r)$ in the hard-core limit, and hence its derivative $e_{1/2}'(r)$ converges to the delta function $\delta(d-r)$. We consider the following quantity, which has dimension of $(\mathrm{time})^{-1}$, for illustration,

$$\langle \Lambda(\boldsymbol{\Gamma})\rangle_{\mathrm{eq}} = -\frac{\zeta}{m}\left\langle \sum_{i,j}{}' \Theta(d-r_{ij}) \right\rangle_{\mathrm{eq}}$$
$$\approx -\epsilon N n \sqrt{\frac{\kappa}{m}} \int d^3\boldsymbol{r}\, g(r)\Theta(d-r). \quad \text{(G9)}$$

From $e_{1/2}'(r)\, y_{1/2}(r) = \sqrt{\beta_{\mathrm{SS}}\kappa}\, \Theta(d-r) g(r)$, we obtain

$$\langle \Lambda(\boldsymbol{\Gamma})\rangle_{\mathrm{eq}} = -\epsilon N n \sqrt{\frac{T_{\mathrm{SS}}}{m}} \sqrt{\frac{\kappa}{T_{\mathrm{SS}}}} \int d^3\boldsymbol{r}\, g(r)\Theta(d-r)$$
$$= -\epsilon N n \sqrt{\frac{T_{\mathrm{SS}}}{m}} \int d^3\boldsymbol{r}\, e_{1/2}'(r) y_{1/2}(r)$$
$$\to -4\pi\epsilon N n \sqrt{\frac{T_{\mathrm{SS}}}{m}} g_0(\varphi) d^2, \quad \text{(G10)}$$

which corresponds to the replacement

$$\sqrt{\frac{\kappa}{m}}\Theta(d-r)\, g(r) \to \sqrt{\frac{T_{\mathrm{SS}}}{m}}\delta(d-r)\, g_0(\varphi). \quad \text{(G11)}$$

In the non-dimensionalized form, it is given by

$$\Theta(1-r^*)\, g(r^*) \to \Xi(r^*)\, g_0(\varphi), \quad \text{(G12)}$$

where

$$\Xi(r^*) \equiv \sqrt{T_{\mathrm{SS}}^*}\, \delta(1-r^*) = \frac{\omega_E^*(T_{\mathrm{SS}}^*)}{4\sqrt{\pi}\, n^* g_0(\varphi)}\delta(1-r^*). \quad \text{(G13)}$$

Appendix H: Molecular dynamics simulation

We briefly describe the methods of the molecular dynamics (MD) simulation we have performed. The unit of mass, length, and time is chosen as m, d, $\sqrt{m/\kappa}$, and we attach $*$ to non-dimenzionalized quantities, e.g. $t^* = t\sqrt{\kappa/m}$. The parameters of the simulation are the volume fraction φ, the shear rate $\dot{\gamma}^*$, the dissipation rate $\epsilon = \zeta/\sqrt{\kappa m}$, and the number of spheres N. The conditions are $N = 2000$, $\epsilon = 0.018375$, $\dot{\gamma}^* = 10^{-3}, 10^{-4}, 10^{-5}$, while φ is varied in the range 0.50 and 0.66.

The governing equation is the Sllod equation for uniformly sheared systems, Eq. (A4). This equation is integrated numerically by the velocity Verlet algorithm. The time step of the calculation is chosen as $\Delta t^* = 0.01$. We start with thermalizing the system at an initial temperature $T_{\rm ini}$ in the absence of shear and dissipation, i.e. by setting $\dot{\boldsymbol{\gamma}} = 0$ and $\boldsymbol{F}^{(\rm vis)} = 0$ in Eq. (A4). After thermalization, we switch on the shear and dissipation simultaneously, and evolve the system until the temperature $T(t) = \sum_{i=1}^N \boldsymbol{p}_i(t)^2/(3Nm)$ is relaxed and starts to fluctuate around a steady value. We regard this behavior as a signal for reaching the steady state, and this steady value is identified as the steady-state temperature, $T_{\rm SS}$. Note that $T_{\rm SS}$ is determined solely by the balance of energy, i.e. $\dot{\gamma}$ and ζ, and is independent of the choice of $T_{\rm ini}$. We extract the relaxation time $\tau_{\rm rel}$ by fitting the relaxation behavior of $T(t)$ with an exponential function, $e^{-t/\tau_{\rm rel}}$.

Then, the ensemble average of the shear stress around the steady state is measured by calculating

$$\left\langle \sigma_{xy}^*(\boldsymbol{\Gamma}) \right\rangle_{\rm SS} = \frac{1}{V^*} \sum_{i=1}^N \left[p_{i,x}^* p_{i,y}^* + y_i^* \left(F_{i,x}^{(\rm el)*} + F_{i,x}^{(\rm vis)*} \right) \right]. \tag{H1}$$

In order to suppress the statistical error, we sum the values of Eq. (H1), sampled at some interval in a single run, and divide the sum by the number of samples after the run is terminated. We have verified that the errors between independent runs are negligible.

[1] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys **68**, 1259 (1996).
[2] C. S. O'Hern, S. A. Langer, A. J. Liu, and S. R. Nagel, Phys. Rev. Lett. **88**, 075507 (2002).
[3] C. S. O'Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, Phys. Rev. E **68**, 011306 (2003).
[4] M. Otsuki and H. Hayakawa, Phys. Rev. E **90**, 042202 (2014).
[5] C. Coulais, A. Seguin, and O. Dauchot, Phys. Rev. Lett. **113**, 198001 (2014).
[6] P. N. Pusey, in *Liquids, Freezing and Glass Transition, Part II*, Les Houches Summer School Proceedings Vol. 51, edited by J.-P. Hansen, D. Levesque, and J. Zinn-Justin (Elsevier, Amsterdam, 1991).
[7] D. J. Durian and D. A. Weitz, in *"Foams" in Kirk-Othmer Encyclopedia of Chemical Technology, 4th ed.*, edited by J. I. Kroschwitz (Wiley, New York, 1994) p. 783.
[8] A. J. Liu and S. R. Nagel, Nature (London) **396**, 21 (1998).
[9] A. Ikeda, L. Berthier, and P. Sollich, Phys. Rev. Lett. **109**, 018301 (2012).
[10] S. Torquato and F. H. Stillinger, Rev. Mod. Phys **82**, 2633 (2010).
[11] G. Parisi and F. Zamponi, Rev. Mod. Phys. **82**, 789 (2010).
[12] A. Liu and S. R. Nagel, Ann. Rev. Cond. Matt. Phys. **1**, 347 (2010).
[13] F. da Cruz, S. Emam, M. Prochnow, J.-N. Roux, and F. Chevoir, Phys. Rev. E **72**, 021309 (2005).
[14] M. Wyart, L. E. Silbert, S. R. Nagel, and T. A. Witten, Phys. Rev. E **72**, 051306 (2005).
[15] P. Olsson and S. Teitel, Phys. Rev. Lett. **99**, 178001 (2007).
[16] T. Hatano, Phys. Rev. E **75**, 060301(R) (2007).
[17] T. Hatano, J. Phys. Soc. Jpn. **77**, 123002 (2008).
[18] M. Otsuki and H. Hayakawa, Prog. Theor. Phys. **121**, 647 (2009).
[19] M. Otsuki and H. Hayakawa, Phys. Rev. E **80**, 011308 (2009).
[20] M. Otsuki and H. Hayakawa, Phys. Rev. E **83**, 051301 (2011).
[21] B. P. Tighe, E. Woldhuis, J. J. C. Remmers, W. van Saarloos, and M. van Hecke, Phys. Rev. Lett. **105**, 088303 (2010).
[22] D. Bi, J. Zhang, B. Chakraborty, and R. P. Behringer, Nature **480**, 355 (2011).
[23] P. Olsson and S. Teitel, Phys. Rev. Lett. **109**, 108001 (2012).
[24] P. Olsson and S. Teitel, Phys. Rev. E **88**, 010301(R) (2013).
[25] E. DeGiuli, G. Düring, E. Lerner, and M. Wyart, Phys. Rev. E **91**, 062206 (2015).
[26] F. Boyer, E. Guazzelli, and O. Pouliquen, Phys. Rev. Lett. **107**, 188301 (2011).
[27] M. Trulsson, B. Andreotti, and P. Claudin, Phys. Rev. Lett. **109**, 118305 (2012).
[28] B. Andreotti, J.-L. Barrat, and C. Heussinger, Phys. Rev. Lett. **109**, 105901 (2012).
[29] E. Lerner, G. Düring, and M. Wyart, Proc. Natl. Acad. Sci. U.S.A. **109**, 4798 (2012).
[30] T. Kawasaki, D. Coslovich, A. Ikeda, and L. Berthier, Phys. Rev. E **91**, 012203 (2015).
[31] M. Otsuki, H. Hayakawa, and S. Luding, Prog. Theor. Phys. Suppl. **184**, 110 (2010).
[32] R. A. Bagnold, Proc. R. Soc. Lond. A **225**, 49 (1954).
[33] J. T. Jenkins and M. W. Richman, Phys. Fluids **28**, 3485 (1985).
[34] J. T. Jenkins and M. W. Richman, Arch. Rat. Mech. Anal. **87**, 355 (1985).
[35] V. Garzó and J. W. Dufty, Phys. Rev. E **59**, 5895 (1999).
[36] J. F. Lutsko, Phys. Rev. E **72**, 021306 (2005).
[37] K. Saitoh and H. Hayakawa, Phys. Rev. E **75**, 021302 (2007).

[38] N. V. Brilliantov and T. Pöschel, *Kinetic Theory of Granular Gases* (Oxford, 2010).
[39] M. Y. Louge, Phys. Rev. E **67**, 061303 (2003).
[40] N. Mitarai and H. Nakanishi, Phys. Rev. Lett. **94**, 128001 (2005).
[41] N. Mitarai and H. Nakanishi, Phys. Rev. E **75**, 031305 (2007).
[42] J. T. Jenkins and D. Berzi, Gran. Matt. **12**, 151 (2010).
[43] S. Chialvo and S. Sundaresan, Phys. Fluids **25**, 070603 (2013).
[44] W. Götze, *Complex Dynamics of Glass-Forming Liquids: A Mode-Coupling Theory* (Oxford, 2009).
[45] W. T. Kranz, M. Sperl, and A. Zippelius, Phys. Rev. Lett. **104**, 225701 (2010).
[46] M. Sperl, W. T. Kranz, and A. Zippelius, Europhys. Lett. **98**, 28001 (2012).
[47] W. T. Kranz, M. Sperl, and A. Zippelius, Phys. Rev. E **87**, 022207 (2013).
[48] J. Talbot, R. D. Wildman, and P. Viot, Phys. Rev. Lett. **107**, 138001 (2011).
[49] A. Gnoli, A. Petri, F. Dalton, G. Pontuale, G. Gradenigo, A. Sarracino, and A. Puglisi, Phys. Rev. Lett. **110**, 120601 (2013).
[50] K. Kanazawa, T. G. Sano, T. Sagawa, and H. Hayakawa, Phys. Rev. Lett. **114**, 090601 (2015).
[51] K. Kanazawa, T. G. Sano, T. Sagawa, and H. Hayakawa, J. Stat. Phys. (to be published); arXiv:1412.2233.
[52] H. Hayakawa and M. Otsuki, Prog. Theor. Phys. **119**, 381 (2008).
[53] K. Suzuki and H. Hayakawa, AIP Conf. Proc. **1542**, 670 (2013), (Powders & Grains 2013).
[54] K. Suzuki and H. Hayakawa, AIP Conf. Proc. **1628**, 457 (2014).
[55] O. Dauchot, G. Marty, and G. Biroli, Phys. Rev. Lett. **95**, 265701 (2005).
[56] M. P. Ciamarra and A. Coniglio, Phys. Rev. Lett. **103**, 235701 (2009).
[57] V. Kumaran, J. Fluid. Mech. **632**, 109 (2009).
[58] D. J. Evans and G. P. Morriss, *Statistical Mechanics of Nonequilibrium Liquids, 2nd ed.* (Cambridge, 2008).
[59] L. D. Landau and E. M. Lifshitz, *Mechanics, 3rd ed.* (Butterworth-Heinemann, 1976).
[60] Refer to the Appendix.
[61] P. Résibois, J. Stat. Phys **2**, 21 (1970).
[62] S. Torquato, Phys. Rev. E **51**, 3170 (1995).
[63] Although the scaling of σ_{xy} must be $(\varphi-\varphi_J)^{1/2}$ if the contact network is unchanged for small strain γ, the contact network should be rearranged under a plane shear, because $\gamma=\dot\gamma t$ is not small even for small $\dot\gamma$ in the steady state in the limit $t\to\infty$. If γ is not small, σ_{xy} satisfies $y_\phi=1$ [4].
[64] R. Seto, R. Mari, J. F. Morris, and M. M. Denn, Phys. Rev. Lett. **111**, 218301 (2013).
[65] H. Katsuragi and D. J. Durian, Nature Physics **3**, 420 (2007).
[66] Y. Takehara, S. Fujimoto, and K. Okumura, Euro. Phys. Lett. **92**, 44003 (2010).
[67] H. Katsuragi and D. J. Durian, Phys. Rev. E **87**, 052208 (2013).
[68] Y. Takehara and K. Okumura, Phys. Rev. Lett. **112**, 148001 (2014).
[69] S. Takada and H. Hayakawa, arXiv:1504.04805.
[70] J. F. Brady, J. Chem. Phys. **99**, 567 (1993).
[71] L. E. Silbert, D. Ertaş, G. S. Grest, T. C. Halsey, D. Levine, and S. J. Plimpton, Phys. Rev. E **64**, 051302 (2001).
[72] L. E. Silbert, J. W. Landry, and G. S. Grest, Phys. Fluids **15**, 1 (2003).
[73] J. J. Brey, J. W. Dufty, and A. Santos, J. Stat. Phys. **87**, 1051 (1997).
[74] J. W. Dufty, A. Baskaran, and J. J. Brey, Phys. Rev. E **77**, 031310 (2008).
[75] A. Baskaran, J. W. Dufty, and J. J. Brey, Phys. Rev. E **77**, 031311 (2008).
[76] S.-H. Chong, M. Otsuki, and H. Hayakawa, Phys. Rev. E **81**, 041130 (2010).
[77] H. Hayakawa and M. Otsuki, Phys. Rev. E **88**, 032117 (2013).
[78] J.-P. Hansen and I. R. McDonald, *Theory of Simple Liquids, 3rd ed.* (Academic Press, London, 2006).
[79] A. Donev, S. Torquato, and F. H. Stillinger, Phys. Rev. E **71**, 011105 (2005).